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CONTENTS

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SEMINUCLEAR EXTENSIONS OF GALOIS FIELDS.*

By D. R. Hughes[1] and Erwin Kleinfeld.[2]

We consider here the problem of obtaining all division rings R (not associative) that are quadratic extensions of a Galois field F, where F is to be contained in the right and middle nuclei of R. This problem was inspired by the discovery of such a division ring with 16 elements ([2]). As a result we obtain a new class of finite division rings and a corresponding class of projective planes (in another paper ([1]) the collineation groups for this class of planes are discussed). We find that every Galois field not a prime field is capable of being extended in this day; if we further restrict R in such a way that F is to be contained in the nucleus of R, then exactly those Galois fields which are themselves quadratic extensions permit such an extension.

Definition. The *left nucleus* of a ring R is the set of all a in R such that $(ax)y = a(xy)$ for all x, y in R. The *middle nucleus* of R is the set of all b in R such that $(xb)y = x(by)$ for all x, y in R. The *right nucleus* of R is the set of all c in R such that $(xy)c = x(yc)$ for all x, y in R. The *nucleus* of R is the intersection of the right, middle and left nuclei.

From now on we shall assume that R is a not associative division ring that is a quadratic extension of a Galois field F, such that F is contained in the right and middle nuclei of R. Then R can be represented as a two-dimensional right vector space over F; let 1, λ be a basis of R over F. All elements of R have the form $x + \lambda y$, where x, y are in F; R will be completely determined once its multiplication is specified. If z is an arbitrary generator of the multiplicative group of non-zero elements of F, then the multiplication in R will be determined once $z\lambda$ and λ^2 are known. For we expand

$$(x + \lambda y)(u + \lambda v) = xu + \lambda(yu) + (x\lambda)v + \lambda(y\lambda)v.$$

If $\lambda^2 = \delta_0 + \lambda\delta_1$ and for any s in F, $s\lambda = s_0 + \lambda s_1$, then

* Received May 10, 1959.

[1] Supported in part by the United States Air Force under Contract No. AF 18(600)-1383.

[2] Supported in part by the United States Army Office of Ordnance Research.

$$(x+\lambda y)(u+\lambda v)=xu+\lambda(yu)+x_0v+\lambda(x_1v)+\lambda[y_0v+\lambda(y_1v)]$$
$$=(xu+x_0v+\delta_0y_1v)+\lambda(yu+x_1v+y_0v+\delta_1y_1v).$$

Let us suppose that $z\lambda=q+\lambda r$ for the element z selected above. Two cases arise naturally: either $r=z$ or $r\neq z$. If $r=z$, then $z\lambda=q+\lambda z$; assume inductively that $z^i\lambda=iz^{i-1}q+\lambda z^i$. Then

$$z^{i+1}\lambda=z(z^i\lambda)=iz^iq+(z\lambda)z^i=iz^iq+z^iq+\lambda z^{i+1}=(i+1)z^iq+\lambda z^{i+1},$$

and so the formula is established. Suppose that F has p^n elements, where p is a prime, and substitute $i=p^n$ in the formula. Since then $z^{p^n}=z$, we have $z\lambda=\lambda z$, and so $q=0$. But then one verifies easily that R is commutative, and that λ is in the nucleus of R. In other words, R must be associative, contrary to assumption; so the case $r=z$ does not arise. We may therefore assume that $z\lambda=q+\lambda r$, where $z\neq r$. Now let $\lambda'=\lambda+q/(r-z)$. It is immediate that $z\lambda'=\lambda' r$ and therefore $z^j\lambda'=\lambda' r^j$. In other words, we could have selected λ in such a way that for every x in F, $x\lambda=\lambda x^\sigma$, where x^σ is in F. Let us examine the mapping σ in more detail. Since $(x+y)\lambda=x\lambda+y\lambda=\lambda x^\sigma+\lambda y^\sigma=\lambda(x^\sigma+y^\sigma)$, we see that $(x+y)^\sigma=x^\sigma+y^\sigma$. Also,

$$(xy)\lambda=x(y\lambda)=x(\lambda y^\sigma)=(x\lambda)y^\sigma=(\lambda x^\sigma)y^\sigma=\lambda(x^\sigma y^\sigma),$$

and so $(xy)^\sigma=x^\sigma y^\sigma$. Since R is a division ring, σ must be one-to-one, and so σ is an automorphism of F.

Multiplication in R is now completely determined by σ, and the elements δ_0 and δ_1 in F. In fact, we have

$$(1)\qquad (x+\lambda y)(u+\lambda v)=(xu+\delta_0y^\sigma v)+\lambda(yu+x^\sigma v+\delta_1y^\sigma v).$$

Two issues remain to be clarified. Namely, under what circumstances is R both not associative and a division ring, and under what circumstances is F contained in the right and middle nuclei of R? We answer the latter question first.

Let $a=x+\lambda y$, $b=u+\lambda v$, $c=w+\lambda z$ be three arbitrary elements of R. Then the associator $(a,b,c)=(ab)c-a(bc)$ may be easily calculated, using (1), and we find that

$$(2)\qquad \begin{aligned}(a,b,c)&=\delta_0v^\sigma z[(x^{\sigma^2}-x)+(\delta_1{}^\sigma y^{\sigma^2}-\delta_1y^\sigma)]\\&\quad+\lambda v^\sigma z[(\delta_0{}^\sigma y^{\sigma^2}-\delta_0y)+(\delta_1{}^\sigma y^{\sigma^2}-\delta_1y^\sigma)+\delta_1(x^{\sigma^2}-x^\sigma)].\end{aligned}$$

If either $v=0$ or $z=0$ in (2), then $(a,b,c)=0$, and so F is certainly contained in the right and middle nucleus of R.

Now we investigate the conditions under which R will be a division ring. Since R is finite, R will be a division ring if and only if R has no divisors

of zero. Suppose $(x+\lambda y)(u+\lambda v)=0$, where not both u and v are zero. From (1) we obtain $xu+\delta_0 y^\sigma v=0$, as well as $yu+x^\sigma v+\delta_1 y^\sigma v=0$. Then the matrix of the two equations involving the variables u and v must be singular. The determinant is easily computed to be $x^{1+\sigma}+\delta_1 xy^\sigma-\delta_0 y^{1+\sigma}$, so we set this equal to zero. If $y=0$, then $x^{1+\sigma}=xx^\sigma=0$, so $x=0$. Therefore assume that $y\neq 0$, and let $w=xy^{-1}$; then $x^{1+\sigma}+\delta_1 xy^\sigma-\delta_0 y^{1+\sigma}=y^{1+\sigma}[w^{1+\sigma}+\delta_1 w-\delta_0]=0$, and so $w^{1+\sigma}+\delta_1 w-\delta_0=0$. We have demonstrated that R is a division ring if and only if

$$w^{1+\sigma}+\delta_1 w-\delta_0=0 \tag{3}$$

has no solution for w in F.

Let us assume that in addition to the previous conditions, F is even contained in the nucleus of R. Putting $y=0$ in (2), we obtain $(a,b,c)=\delta_0 v^\sigma z(x^{\sigma^2}-x)+\lambda v^\sigma z(x^{\sigma^2}-x^\sigma)\delta_1=0$. Consequently $\delta_0 v^\sigma z(x^{\sigma^2}-x)=0$. If $\delta_0{}^*=0$, then $\lambda^2=\lambda\delta_1$ implies that R has divisors of zero, contrary to assumption, and so $\delta_0\neq 0$. But then $x^{\sigma^2}=x$ for all x in F, so that $\sigma^2=I$, the identity mapping. Furthermore, $v^\sigma z(x^{\sigma^2}-x^\sigma)\delta_1=0$, so either $\sigma=I$ or $\delta_1=0$. But putting $\sigma=I$ in (1) we see that R would then be associative. So if R is a not associative division ring and F is in its nucleus, then $\sigma^2=I$ and $\delta_1=0$.

Suppose now that R is an associative division ring and $\sigma\neq I$. Then $\sigma^2=I$ and $\delta_1=0$. Substituting these values in (2), we discover that $\lambda v^\sigma yz(\delta_0{}^\sigma-\delta_0)=0$; consequently $\delta_0{}^\sigma=\delta_0$. Now if E is defined to be the fixed field of σ in F, then δ_0 is in E. As w ranges over F, $w^{1+\sigma}$ ranges over all of E, since σ has order two. Thus no matter how δ_0 is chosen, since $\delta_1=0$, (3) will have a solution for w in F, and hence R will not be a division ring. We have reached a contradiction. So if R is a division ring with F in the right and middle nuclei of R and R a quadratic extension of F, then R is associative if and only if $\sigma=I$.

We summarize the results obtained so far in the following theorems.

THEOREM 1. *Let R be a not associative division ring which is a quadratic extension of a Galois field F, and suppose F is contained in the right and middle nuclei of R. Then R must be isomorphic to a ring S constructed as follows: Let S be a vector space of dimension 2 over F, having basis 1, λ and multiplication defined by $(x+\lambda y)(u+\lambda v)=(xu+\delta_0 y^\sigma v)+\lambda(yu+x^\sigma v+\delta_1 y^\sigma v)$, where σ is an arbitrary non-identity automorphism of F and δ_0, δ_1 in F are subject only to the condition that $w^{1+\sigma}+\delta_1 w-\delta_0=0$ have no solution for w in F. Conversely, given F, σ, δ_0, δ_1, satisfying the above conditions, then S will satisfy the conditions on R.*

THEOREM 2. *Let R be a not associative division ring which is a quadratic extension of a Galois field F, and suppose F is contained in the nucleus of R. Then R must be isomorphic to one of the rings S of Theorem* 1 *with the additional stipulation that $\sigma^2 = I$ and $\delta_1 = 0$. Conversely, all such S satisfy the conditions on R.*

At this point the following question arises: given a Galois field F, does there exist an extension R satisfying Theorems 1 and 2? First we consider Theorem 1. In order to obtain such an R we need to produce an automorphism $\sigma \neq I$ of F and elements δ_0, δ_1 in F such that (3) has no solution in F. Suppose F has p^n elements, p a prime. If $n = 1$, $\sigma \neq I$ cannot exist, so R does not exist either. Assume that $n > 1$; two cases arise. If $p > 2$, choose $\delta_1 = 0$, and $x^\sigma = x^p$. Since $(-1)^{1+\sigma} = (-1)^{1+p} = 1 = (1)^{1+\sigma}$, there must exist an element not of the form $w^{1+\sigma}$, for w in F; let δ_0 be such an element. Then (3) is not satisfied by any w in F. If $p = 2$, choose $\delta_1 = 1$, $x^\sigma = x^2$. Since the mapping which sends x onto $x^3 + x$ send both 0 and 1 onto 0, there exists an element in F which is not of the form $x^3 + x$; choose δ_0 to be such an element. Again (3) cannot be satisfied by any w in F. Thus as long as F is not a prime field, an extension of F as described in Theorem 1 always exists.

A similar argument applies if an extension of F satisfying the hypotheses of Theorem 2 is to exist. In that case F must have an automorphism of order 2. So if F has order p^n, n must be even; that is, F must itself be a quadratic extension of a Galois field to begin with. Conversely, if F has p^{2k} elements then the extension described in Theorem 2 will always be possible by a suitable choice of σ and δ_0.

We conclude with the remark that while the construction of R using (1) will yield division rings when F is an infinite field, not all division rings which are quadratic extensions of such an F, with F in the middle and right nuclei, need be of the that form. In particular, Theorems 1 and 2 are no longer valid, and we have omitted the discussion of the infinite case.

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REGULAR MAPPINGS WHOSE INVERSES ARE 3-CELLS.*[1]

By Mary-Elizabeth Hamstrom.

1. **Introduction.** In our paper [6] Eldon Dyer and I introduced the notion of a completely regular mapping, f, of the metric space X onto a metric space Y (see Definitions 2.1 and 2.2) and were able to prove that under certain additional hypotheses on X, Y, and the inverses under f, (X, f, Y) is a locally trivial fibre space. Under some conditions, f is the projection mapping of a direct product. In [6] and [7] it was shown that if f is a 0-regular mapping of a metric space X onto a metric space Y and M is a compact 2-manifold with boundary such that each inverse under f is homeomorphic to M, then f is completely regular. Thus it may be proved that if X is complete and Y has finite covering dimension, then (X, f, Y) is a locally trivial fibre space and if Y is locally compact, separable, and contractible, then X is homeomorphic to the direct product $Y \times M$, f corresponding to the projection map of $Y \times M$ onto Y. The purpose of the present paper is to extend some of these results to the case where f is homotopy 2-regular and M is a compact 3-manifold with boundary which is imbeddable in E^3. A later paper will consider more general 3-manifolds [8].

Section 2 states some definitions and proves some lemmas concerning the convergence of the boundaries of the 3-manifold inverses under regular mappings. The principal result of Section 3 is Theorem 3.13, which states that if f is a homotopy 2-regular mapping of a metric space X onto a metric space Y such that each inverse under f is a 3-cell, M, then f is completely regular. The proof involves complicated constructions and push-pull arguments, which the reader may find easier to follow if he first reads Definition 2.7 and then the statements of the lemmas and theorems of Section 3 in decreasing order. Section 4 involves an induction argument on the number of elements in a cellular decomposition of M to yield Theorem 4.5, which extends Theorem 3.13 to the case where M is a compact 3-manifold with

* Received February 10, 1959.

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boundary and is imbeddable in E^3. Section 5 deals with the space of homeomorphisms on a 3-manifold and extends the results mentioned in the first paragraph above to completely regular mappings whose inverses are 3-manifolds with boundary which are imbeddable in E^3. Section 6 considers some slight relaxations in the hypotheses of some previous theorems.

2. Homotopy regular mappings whose inverses are 3-manifolds.

Definition 2.1. A proper mapping f of a metric space X onto a metric space Y is *homotopy n-regular* (*h-n-regular*) provided that it is true that f is open and if x is a point of X and ϵ is a positive number, then there is a positive number δ such that every mapping of a k-sphere, $k \leqq n$, into $S(x,\delta) \cap f^{-1}(y)$, $y \in Y$, is homotopic to 0 in $S(x,\epsilon) \cap f^{-1}(y)$. (A proper map is one for which inverses of compact sets are compact. The symbol $S(x,\epsilon)$ denotes the set of points with distance from x less than ϵ.)

Definition 2.2. The mapping f is said to be *completely regular* provided that if $\epsilon > 0$ and $y \in Y$, then there is a $\delta > 0$ such that $d(y,y') < \delta$, $y' \in Y$, implies that there is a homeomorphism of $f^{-1}(y)$ onto $f^{-1}(y')$ which moves no point as much as ϵ (i.e. an ϵ-homeomorphism).

In the sequel, a sequence $x_1, x_2, \cdots$ may sometimes be denoted by the symbol $\{x_i\}_i$ or, if no ambiguity results, by $\{x_i\}$. A point P with the property that every open set containing P intersects all but a finite number of the elements of the sequence $\{x_i\}$ will be called a *sequential limit point* of $\{x_i\}$.

Definition 2.3. If f is an h-n-regular (completely regular) mapping of a compact space X onto a space Y and Y consists of the points of the sequence $y_0, y_1, y_2, \cdots$ which converges to y_0, then the sequence $\{f^{-1}(y_i)\}$ is said to *converge h-n-regularly* (*completely regularly*) to $f^{-1}(y_0)$.

In the remainder of this section, $M_0, M_1, M_2, \cdots$ will denote the elements of a sequence of compact 3-manifolds with boundary converging h-2-regularly to M_0, the union of whose elements is a compact metric space. It will further be supposed that the boundaries, $K_0, K_1, K_2, \cdots$, of these 3-manifolds are mutually homeomorphic and that each M_i is polyhedrally imbeddable in E^3. The space $\cup M_i$ is clearly finite dimensional and hence may be considered as a subset of some Euclidean space, E^{n-3}. Furthermore, it follows from a result of Klee ([10], 3.3, p. 36) that $\cup M_i$ may be so imbedded in E^n that M_0 is a polyhedral subset of a 3-dimensional hyperplane of E^n.

The 3-manifold M_i may be triangulated by means of a triangulation Γ_i.

In each M_i, polyhedra, polygons, et cetera will be defined relative to Γ_i. No particular relationships among the Γ_i's will be assumed. Distances will be ordinary distances in E^n. *Unless it is explicitly stated otherwise, any subset of M_i that is referred to will be polyhedral.*

If M is a manifold with boundary, the notation bdry M and int M will be used to denote the sets of boundary points and non-boundary points of M respectively.

The content of the first two lemmas is probably known, but they are included for completeness.

LEMMA 2.4. *If M is a compact 2-manifold with boundary, which may be empty, and ϵ is a positive number, there is a positive number δ such that each mapping of M into itself which moves no point as much as δ is ϵ-homotopic to the identity map.*

Indication of proof. Dente by $J_1, J_2, \cdots, J_m$ the boundary simple closed curves of M and by $A_1, \cdots, A_m$ mutually exclusive annuli in M such that for each i, A_i is bounded by J_i and a simple closed curve J_i' which is $\epsilon/2$-isotopic to J_i in A_i. For sufficiently "narrow" A_i and small δ, each δ-mapping f of M into itself carries J_i into $A_i - J_i'$ and is thus ϵ-homotopic relative to $f^{-1}(M - A_i)$ to a mapping which, when restricted to J_i, is the identity. The proof of Lemma 2.4 may thus be restricted to the consideration of mappings leaving bdry M pointwise fixed.

Let G be a cellular decomposition of M in the sense that G is a finite collection $g_1, g_2, \cdots, g_k$ of discs such that $\cup g_i = M$ and if $g_i \cap g_j$ exists, it is an arc. The proof proceeds by induction on k. The lemma is clearly true if $k = 1$. Suppose it to be true for all compact 2-manifolds with boundary which have a cellular decomposition with fewer than k elements and that $G: g_1, \cdots, g_k$ is a cellular decomposition of M. It may be assumed that each component of $g_i \cap$ bdry M is an arc. If c is an element of G, $\mathrm{cl}(M - c)$ is a compact 2-manifold with boundary which has a cellular decomposition with fewer than k elements. A slight modification of the content of the previous paragraph may be used to prove that for sufficiently small δ, each δ-mapping of M onto itself leaving bdry M pointwise fixed is $\epsilon/2$-homotopic to a mapping leaving bdry $M \cup$ bdry c pointwise fixed. The induction hypothesis may now be used to prove the lemma.

LEMMA 2.5. *If f is a mapping of a compact 2-manifold M onto a compact 2-manifold N which is not homotopic to a mapping carrying M into a proper subset of N and A is an annulus in N, then some simple closed curve,*

J, in $f^{-1}(A)$ is mapped essentially into A (i.e. $f|J$ is not homotopic to 0 in A). As a consequence, for sufficiently small ϵ, if f^* is a piecewise linear ϵ-approximation to f and z is a non-trivial 1-cycle carried by J, then $f^*(z)$ does not bound in A.

Proof. Let A' and A'' denote annuli such that $A'' \subset \operatorname{int} A'$, $A' \subset \operatorname{int} A$ and each circles those in which it is contained, let K denote the closure of $A - A'$ and let t denote an arc in A' which lies except for its endpoints in $\operatorname{int} A'$ and whose endpoints lie in different components of K. Suppose that no simple closed curve in $f^{-1}(A)$ has the required property. If U is a component of $f^{-1}(A) - f^{-1}(K)$ the closure of whose image lies in $A' - A''$, then f is homotopic relative to $M - U$ to a mapping carrying U into K and $f(U)$ remains in $A' - A''$ during the homotopy. Thus it may be assumed that $f^{-1}(A) - f^{-1}(K)$ has only a finite number of components. If U is one such component, there is a 2-manifold with boundary, N', lying in U such that the closure of $f(U - N')$ lies in $A' - A''$. Since $f | \operatorname{bdry} N'$ is homotopic to 0 in A, there is a 2-manifold with boundary, N'', in N' which is homeomorpic to N' and whose boundary is in a small neighborhood of bdry N' in $f^{-1}(A' - A'')$ and there is a homotopy of f relative to $M - N'$ into a mapping g carrying bdry N'' into two points, $f(N' - N'')$ remaining in $A' - A''$ during the homotopy. The mapping g is, in turn, homotopic relative to $M - U$ to a mapping f_1 which carries U into $K \cup t$, $f(U - N')$ remaining $A' - A''$ during the homotopy. Repeat this process until a mapping f_k is obtained which is homotopic to f and carries M into $(M - A') \cup t$. This contradiction proves the lemma.

LEMMA 2.6. *If ϵ is a positive number and H_0 is a polyhedron in M_0, then there is an integer N such that if $i > N$, then there is a piecewise linear ϵ-mapping of H_0 into* $\operatorname{int} M_i$.

Proof. The compactness of $\cup M_i$ and the h-2-regularity of the convergence of $\{M_i\}$ to M_0 imply the existence of numbers $\delta_0, \delta_1, \delta_2, \delta_3 = \epsilon$, $0 < \delta_1 < \delta_2 < \delta_3$, such that every mapping of a k-sphere, $k \leqq 2$, into a subset of M_i of diameter less than δ_k is homotopic to 0 on a subset of M_i of diameter less than $\delta_{k+1}/(k+3)$. Let T_0 be a triangulation of H_0 of mesh less than $\delta_0/3$, the simplices of T_0 being in some subdivision of Γ_0. (The mesh of T_0 is the maximum of the diameters of the simplices of T_0.) There is an integer N such that if $i > N$, then there is a $\delta_0/3$-homeomorphism g_i^0 of the 0-skeleton, T_0^0 of T_0 into $\operatorname{int} M_i$. If a and b are the vertices of a 1-simplex, s_0^1, of T_0, $d(g_i^0(a), g_i^0(b)) < \delta_0$. Consequently, there is an arc

s_i^1 in M_i of diameter less than $\delta_1/3$ whose endpoints are $g_i^0(a)$ and $g_i^0(b)$. This arc may be taken to be polygonal and a subset of $\operatorname{int} M_i$. Also, small changes in the s_i^1's can be made so that no two of them intersect except, possibly, at their endpoints.

If each mapping $g_i^0 \mid (a \cup b)$ is extended to a homeomorphism of s_0^1 onto s_i^1, g_i^0 is extended to a piecewise linear homeomorphism g_i^1 of the 1-skeleton T_0^1 of T_0 into $\operatorname{int} M_i$. Clearly g_i^1 is a δ_1-homeomorphism and if ∂s_0^2 is the boundary of a 2-simplex s_0^2 of T_0, $\operatorname{diam} g_i^1(\partial s_0^2) < \delta_1$. Thus $g_i^1 \mid \partial s_0^2$ can be extended to a mapping of s_0^2 onto a subset s_i^2 of M_i of diameter less than $\delta_2/4$, which, by the simplicial approximation theorem, may be assumed to be piecewise linear. Small changes in s_i^2 may be made, if necessary, so that it lies in $\operatorname{int} M_i$. In this way, there is obtained a piecewise linear extension, g_i^2 of g_i^1 a δ_2-mapping of the 2-skeleton, T_0^2, of T_0 into $\operatorname{int} M_i$ with the property that if ∂s_0^3 is the boundary of a 3-simplex s_0^3 of T_0, then $\operatorname{diam} g_i^2(\partial s_0^3) < \delta_2$.

It follows as above that $g_i^2 \mid \partial s_0^3$ can be extended to a piecewise linear ϵ-mapping of s_0^3 into a subset s_i^3 of $\operatorname{int} M_i$. This defines a piecewise linear ϵ-mapping of H_0 into $\operatorname{int} M_i$.

A *regular neighborhood in M_i* of a polyhedron P_i is a closed neighborhood of which P_i is a (strong) deformation retract and whose boundary is a compact polyhedral 2-manifold (not necessarily connected and with boundary if P_i intersects K_i). It is known that every polyhedron in M_i has such a neighborhood (Whitehead, see [5] and [16]). A *regular neighborhood in E^n* of a polyhedron P in M_0 is a closed neighborhood U of which P and $U \cap M_0$ are deformation retracts, which is such that $U \cap M_0$ is a regular neighborhood of P in M_0. The mappings of U onto P and $U \cap M_0$ resulting from the deformation will be called the natural mappings of U onto P and $U \cap M_0$. For each positive number ϵ there exist such neighborhoods U for which no point is moved as much as ϵ during the deformations of U on $U \cap M_0$ and $U \cap M_0$ on P. Such neighborhoods will be called regular ϵ-neighborhoods.

If J is a subpolyderon of P, above, U' is a regular ϵ-neighborhood of J in E^n and r and q are the deformations of U into $U \cap M_0$ and $U \cap M_0$ into P, then U' is said to be *consistently imbedded* in U provided that (1) $U' \cap M_0$ and $(U - U') \cap M_0$ are deformation retracts of U' and $U - U'$ under r, (2) $U' \cap P$ and $(U - U') \cap P$ are deformation retracts of $U' \cap M_0$ and $(U - U') \cap M_0$ under q and (3) J is a deformation retract of $U' \cap P$.

Definition 2.7. If H_0 is a compact polyhedral p-manifold, $p = 1, 2$, in M_0 and $\{H_i\}$ is a sequence of compact polyhedral p-manifolds converging to

H_0 such that for each i, H_i lies in M_i, then the sequence $\{H_i\}$ is said to *converge strongly* to H_0 if it is true that for each regular neighborhood in E^n, V_0, of H_0 and for sufficiently large i, the p-cycles carried by H_i (mod the integers) fail to bound in $V_0 \cap M_i$. If $\{D_i\}$ is a sequence of discs (annuli) converging to the disc (annulus) D_0 in M_0 and for each i, $D_i \subset M_i$, then the sequence $\{D_i\}$ is said to converge strongly to D_0 if the sequence $\{\text{bdry}\, D_i\}$ converges strongly to bdry D_0.

All homology in this paper will be taken modulo the integers.

LEMMA 2.8. *The sequence $K_1, K_2, \cdots$ converges to a subset of K_0.*

Proof. Denote by C_0 a component of K_0 and by U_0 a regular neighborhood of C_0 in E^n, p the natural mapping. Presume that $U_0 \cap (K_0 - C_0) = 0$. It follows from Lemma 2.6 that there are a sequence $\epsilon_1, \epsilon_2, \cdots$ of positive numbers converging to 0 and a sequence $g_1, g_2, \cdots$ of mappings such that for each i, g_i is a piecewise linear ϵ_i-mapping of C_0 into $U_0 \cap \text{int}\, M_i$. For each i, a regular neighborhood V_i of $g_i(C_0)$ may be constructed in $U_0 \cap \text{int}\, M_i$ in such a way that the sequence $\{V_i\}$ converges to C_0. If V_i fails to separate M_i, then, since $V_i \cap K_i = 0$, the boundary of V_i has only one component and the 2-cycle carried by $g_i(C_0)$ into which a fundamental 2-cycle γ_0 of C_0 is mapped by g_i bounds in V_i. If V_i does separate M_i, it follows from the h-2-regularity of the convergence of $\{M_i\}$ to M_0 that for sufficiently large i all but one of the components of $M_i - V_i$ lies in U_0. Denote by V_i' the union of V_i and these components and by C_i' its boundary. If none of these components intersects K_i, then C_i' is a compact, connnected 2-manifold and the 2-cycle $g_i(\gamma_0)$ bounds in V_i'. In any case, $g_i(\gamma_0)$ bounds in $U_0 \cap \text{int}\, M_i$ and thus $p_i g_i(\gamma_0)$ bounds in C_0, where p_i is a piecewise linear ϵ_i-approximation to $p \mid V_i'$. It follows from Lemma 2.4 that for sufficiently large i, $p_i g_i$ is homotopic to the identity mapping of C_0 into itself and thus carries γ_0 into a non-bounding cycle. Since $g_i(\gamma_0)$ bounds in V_i', this is a contradiction. Hence, for sufficiently large i, V_i separates M_i and V_i' contains a component C_i of K_i. The sequence $\{C_i\}$ converges to a subset of C_0. Since this argument holds for each component of K_0 and K_i has the same number of components as K_0, the lemma is proved.

In the next several lemmas, C_0 denotes a component of K_0 and $C_1, C_2, \cdots$ denote components of $K_1, K_2, \cdots$ converging to a subset of C_0. If U_0 is a regular neighborhood of C_0 in E^n not intersecting $K_0 - C_0$, then for sufficiently large i, C_i is the only component of K_i in U_0 and is one of the two components of bdry V_i', the other being denoted by C_i' (V_i' is as defined above).

LEMMA 2.9. *The sequence $\{C_i\}$ converges strongly to C_0.*

Proof. Let p be the natural mapping of U_0 onto C_0 and p_i a piecewise linear ϵ_i-approximation to $p \mid V_i'$, g_i and ϵ_i being defined as in the proof of Lemma 2.8. If γ_i and γ_i' are fundamental 2-cycles carried by C_i and C_i', γ_i is homologous to γ_i' in V_i' (γ_i' being taken with appropriate orientation) and thus $p_i(\gamma_i)$ is homologous to $p_i(\gamma_i')$ in C_0. Hence if γ_i bounds in $U_0 \cap M_i$, then $p_i(\gamma_i)$ and $p_i(\gamma_i')$ are homologous to 0 in C_0. However, $g_i(\gamma_0)$ is a linear combination of γ_i and γ_i'. Thus $p_i g_i(\gamma_0)$ bounds in C_0, contradicting the fact that for sufficiently large i, $p_i g_i$ is homotopic to the identity map of C_0 onto itself and thus carries the non-bounding cycle γ_0 into a non-bounding cycle. Therefore, γ_i fails to bound in $U_0 \cap M_i$.

If a subsequence $\{C_{n_i}\}_i$ of $\{C_i\}$ converges to a proper subset of C_0, then for sufficiently large i, $p_{n_i}(\gamma_{n_i})$ bounds in C_0 and a contradiction follows as above. This completes the proof that the convergence of $\{C_i\}$ to C_0 is strong.

LEMMA 2.10. *If J_0 is a simple closed curve in C_0, there is a sequence $J_1, J_2, \cdots$ of simple closed curves converging strongly to J_0 such that for each i, J_i is a subset of C_i.*

Proof. Let W_0 denote a regular neighborhood of J_0 in E^n consistently imbedded in U_0. If $p \mid C_i$ is homotopic to a map of C_i onto a proper subset of C_0, then $p_i(\gamma_i)$ bounds in C_0, which the proof of Lemma 2.9 has demonstrated to be false. Thus Lemma 2.5 may be applied to yield Lemma 2.10.

LEMMA 2.11. *If, for sufficiently large i, the fundamental* 1*-cycle, z_i, carried by J_i bounds in C_i, then z_0, the fundamental* 1*-cycle carried by J_0, bounds in C_0.*

Proof. For sufficiently large i, a piecewise linear approximation to $p \mid C_i$ carries z_i into a bounding cycle in C_0 which is a multiple of z_0. Thus z_0 bounds in C_0.

LEMMA 2.12. *If z_0 bounds in C_0, then for sufficiently large i, z_i bounds in C_i.*

Proof. Denote by $z_{01}, \cdots, z_{0h}$ a linearly independent system of cycles generating the first homology group, $H^1(C_0)$ of C_0, z_{0j} being carried by a simple closed curve J_{0j} which does not meet J_0, and for each j, let $\{J_{ij}\}_i$ be a sequence of simple closed curves converging strongly to J_{0j}, J_{ij} lying in C_i and carrying a fundamental 1-cycle z_{ij}. A modification of the proof of Lemma 2.11 demonstrates that for sufficiently large i, the z_{ij} are linearly independent and thus that the first Betti number, $p^1(C_i)$, of C_i is not less than $p^1(C_0)$.

However, K_i is homeomorphic to K_0 and this argument can be applied to each component of K_0. Thus $p^1(C_i) = p^1(C_0)$ for sufficiently large i and C_i is homeomorphic to C_0. If z_i does not bound in C_i then a multiple of z_i carried by J_i is a linear combination of $z_{i1}, \cdots, z_{ih}$, $cz_i = \sum c_j z_{ij}$, where not all the c_j are 0. Thus, applying the proof of Lemma 2.11, a multiple of z_0 is a (non-trivial) linear combination of $z_{01}, \cdots, z_{0h}$. This is impossible. Thus, for sufficiently large i, z_i bounds in C_i.

Lemma 2.13. *If B_0 is a disc with boundary J_0 such that $B_0 - J_0 \subset \operatorname{int} M_0$ and $J_0 \subset K_0$, then there is a sequence $B_1, B_2, \cdots$ of discs, with boundaries $J_1, J_2, \cdots$, converging strongly to B_0 such that for each i, $B_i - J_i \subset \operatorname{int} M_i$ and $J_i \subset K_i$.*

Proof. Denote by ϵ a positive number and by U_0 and W_0 regular ϵ-neighborhoods of B_0 and J_0 in E^n, W_0 being consistently imbedded in U_0 and p and q denoting the natural mappings of U_0 and W_0 onto B_0 and J_0. There are positive numbers δ and $\delta' < \delta$ such that every (singular) 1-sphere in M_i of diameter less than δ bounds a (singular) 2-cell in M_i of diameter less than $\epsilon/2$ and every 0-phere in M_i of diameter less than δ' bounds a (singular) 1-cell in M_i of diameter less than $\delta/4$.

It follows from Lemma 2.10 that there is a sequence $\{J_i\}$ of simple closed curves converging strongly to J_0 such that for each i, J_i lies in K_i. There is a simple closed curve J_0' bounding a disc B_0' which is $\delta'/16$-homeomorphic to B_0 and lies in $\operatorname{int} B_0$. For sufficiently large i, $q \mid J_i$ is a $\delta'/16$-mapping q_i of J_i onto J_0 and there is a piecewise linear $\delta'/16$-mapping, g_i, of B_0' into $\operatorname{int} M_i \cap U_0$. Therefore there exists a piecewise linear $\delta'/4$-mapping h_i of J_i onto $g_i(J_0')$. Let $P_{i0}, \cdots, P_{im}$ be a sequence of points of J_i in that order such that the diameter of each component of $J_i - \cup P_{ij}$ is less than $\delta'/4$. There is an arc t_{ij} in M_i of diameter less than $\delta/4$ whose endpoints are P_{ij} and $h_i(P_{ij})$ and which may be constructed so as to meet K_i only in P_{ij}. The (singular) closed curve $t_{ij} \cup t_{ij-1} \cup P_{ij-1}P_{ij} \cup h_i(P_{ij-1}P_{ij})$ has diameter less than δ and consequently bounds a (singular) 2-cell B_{ij} of diameter less than $\epsilon/2$ which lies in $U_0 \cap M_i$ and may be constructed so as to meet K_i only in the arc $P_{ij-1}P_{ij}$ of J_i. The curve $h_i(J_i)$ is contractible in $g_i(B_0')$ and therefore the (singular) 2-cells B_{ij}, $j = 1, \cdots, m$, and $g_i(B_0')$ may be fitted together to form a singular 2-cell in $U_0 \cap M_i$ which is bounded by J_i, meets K_i only in J_i and has no singularities on its boundary. Hence it follows from Dehn's Lemma [12] that J_i bounds a non-singular 2-cell in $U_0 \cap M_i$ which meets K_i only in J_i. Since the sequence $\{J_i\}$ converges strongly to J_0 and U_0 is arbitrary, the existence of the required sequence is demonstrated.

THEOREM 2.14. *The sequence $\{K_i\}$ converges to K_0 h-1-regularly.*

Proof. Consider a component C_0 of K_0 and a sequence $\{C_i\}$ of components of $K_1, K_2, \cdots$ converging strongly to C_0 and all homeomorphic to C_0. Denote by P_0 a point of C_0, by J_0 a simple closed curve bounding a disc A_0 in C_0 whose interior contains P_0 and by B_0 the closure of $C_0 - A_0$. Let E_0 be a disc in M_0 such that $K_0 \cap E_0 = \operatorname{bdry} E_0 = J_0$. It follows from Lemma 2.13 that there are a sequence of simple closed curves $J_1, J_2, \cdots$ converging strongly to J_0 and a sequence $E_1, E_2, \cdots$ of discs converging strongly to E_0 such that for each i, $E_i \cap K_i = E_i \cap C_i = \operatorname{bdry} E_i = J_i$ and $E_i \subset M_i$. It follows readily from Lemma 2.12 that for each i, J_i separates C_i into two sets A_i and B_i whose closures are 2-manifolds with boundary and E_i separates M_i into two sets U_i and V_i, U_i bounded by $E_i \cup A_i$, V_i bounded by $E_i \cup B_i$. It follows from the 2-regularity of the convergence of $\{M_i\}$ to M_0 that the A_i and B_i may be so named that $\{A_i\}$ converges to A_0 and $\{B_i\}$ converges to B_0. Thus, since B_0 contains all the handles of C_0, it follows from Lemma 2.11 and the proof of Lemma 2.12 that B_i contains all the handles of C_i, for sufficiently large i. Therefore A_i is a disc.

If ϵ is a positive number, choose A_0 to lie in $S(P_0,\epsilon)$ and δ to be such that $S(P_0,\delta) \cap C_i \subset A_i$. Since for sufficiently large i, $A_i \subset S(P_0,\epsilon)$ and $B_i \subset C_i - (C_i \cap S(P_0,\delta))$, it follows that every i-sphere $(i = 0, 1)$ in $S(P_0,\delta) \cap C_i$ bounds an $(i+1)$-cell in $S(P^0,\epsilon) \cap C_i$ and the theorem is proved.

COROLLARY. *The sequence $K_1, K_2, \cdots$ converges to K_0 completely regularly.*

Proof. This is a direct consequence of Lemma 3 of [7] mentioned in the introduction.

LEMMA 2.15. *If S_0 is a 2-sphere bounding a 3-cell A_0 in $\operatorname{int} M_0$, then there is a sequence $S_1, S_2, \cdots$ of 2-spheres converging strongly to S_0 such that for each i, S_i bounds a 3-cell A_i in M_i and the sequences $\{A_i\}$ and $\{M_i - A_i\}$ converge to A_0 and $\operatorname{cl}(M_0 - A_0)$.*

Proof. Let U denote a regular ϵ-neighborhood of S_0 in $E^n - K_0$ and U' and V regular ϵ-neighborhoods of S_0 in $E^n - K_0$ such that $V \subset \operatorname{int} U$, $U \subset \operatorname{int} U'$, V is a deformation retract of U and U is a deformation retract of U'. Let p denote the natural mapping of U' onto S_0. There exist, as a consequence of Lemma 2.6, a sequence $\{\delta_i\}$ of positive numbers converging to 0 and sequences $\{g_i\}$ and $\{g_i'\}$ such that for each i, g_i is a piecewise-linear

δ_i-mapping of M_0 into $\operatorname{int} M_i$ and g_i' is a piecewise-linear δ_i-mapping of M_i into $\operatorname{int} M_0$. For sufficiently large i, $\operatorname{int} U \cap M_i$ is a 3-manifold in M_i which contains $g_i(V)$, $g_i'g_i(V)$ is in $\operatorname{int} U$, $g_i'(U \cap M_i)$ is in $\operatorname{int} U'$, and $g_i'(K_i)$ is in $(M_0 - A_0) - ((M_0 - A_0) \cap U')$.

Suppose that $g_i \mid S_0$ is homotopic to 0 in $\operatorname{int} U \cap M_i$ for infinitely many values of i, so that $g_i \mid S_0$ can be extended to a mapping G_i of A_0 into $U \cap M_i$. Then $g_i'G_i$ is a mapping of A_0 into $\operatorname{int} U'$, which implies that $g_i'g_i \mid S_0$ is homotopic to 0 in U', which, for sufficiently large i and small δ_i, is impossible. Hence, for sufficiently large i, $g_i \mid S_0$ is not homotopic to 0 in $U \cap M_i$. It follows from the work of C. D. Papakyriakopoulos [12] (the Sphere Problem) that arbitrary small neighborhoods of $g_i(S_0)$ contain polyhedral 2-spheres, S_i, which are not contractible in $U \cap M_i$. Since this construction can be made for each ϵ, the existence of a sequence $\{S_i\}$ of 2-spheres converging strongly to S_0 is demonstrated. If a non-trivial 2-cycle γ_i carried by S_i bounds in $U \cap M_i$, then, since M_i is imbeddable in E^3, S_i bounds a 3-cell in $U \cap M_i$ and is thus contractible therein, which is impossible.

The 2-regularity of the convergence of $\{M_i\}$ to M_0 implies that for sufficiently large i, one of the components of $M_i - S_i$ contains K_i. The closure of the other is, therefore, a 3-cell, A_i. The 2-regulatory also implies that every point of $M_0 - A_0$ is a sequential limit point of $\{M_i - A_i\}$ and no subsequence of $\{A_i\}$ converges to a subset of U or has a sequential limit point in $M_0 - A_0$. Thus $\{A_i\}$ converges to A_0 and $\{M_i - A_i\}$ converges to $\operatorname{cl}(M_0 - A_0)$, which was to be proved.

LEMMA 2.16. *The notation being that of Lemma 2.15, if T_0 is a torus in $\operatorname{int} A_0$ with interior U_0, then there is a sequence $T_1, T_2, \cdots$ of compact 2-manifolds converging strongly to T_0 such that for each i, T_i lies in $\operatorname{int} A_i$ with interior U_i and the sequences $\{U_i\}$ and $\{M_i - U_i\}$ converge to $T_0 \cup U_0$ and $\operatorname{cl}(M_0 - U_0)$.*

Proof. Let $\epsilon_1, \epsilon_2, \cdots$ be a sequence of positive numbers converging to 0 and $g_1, g_2, \cdots$ a sequence of mappings such that for each i, g_i is a piecewise linear ϵ_i-mapping of T_0 into $\operatorname{int} A_i$. There is a regular neighborhood V_0 of T_0 in E^n such that $V_0 \cap M_0 \subset \operatorname{int} A_0$ and for each i, there is a regular neighborhood V_i of $g_i(T_i)$ in $\operatorname{int} A_i$. The V_i may be so selected that $\{V_i\}$ converges to T_0. Denote the natural mapping of V_0 on T_0 by p.

The h-2-regularity of the convergence of $\{M_i\}$ to M_0 implies that for sufficiently large i, all but at most two of the components of $M_i - V_i$ lie in V_0. Denote the union of V_i and the components of $M_i - V_i$ in V_0 by N_i and the boundary of the component of $M_i - N_i$ containing K_i by T_i. If

$M_i - N_i$ were connected, a fundamental 2-cycle, γ_i, carried by T_i would bound in $V_0 \cap M_i$. Hence $g_i(\gamma_0)$, γ_0 being a fundamental 2-cycle carried by T_0, would bound in $V_0 \cap M_i$, since it lies in N_i. Hence $p_i g_i(\gamma_0)$ would bound in T_0, where p_i is a piecewise linear ϵ_i-approximation to $p \mid N_i$. But it follows from Lemma 2.4 that for sufficiently large i and small ϵ_i, $p_i g_i$ is homotopic to the identity map, so $p_i g_i(\gamma_0)$ does not bound. This contradiction implies that $M_i - N_i$ has two components, the one containing K_i and bounded *by* T_i, the other denoted by U_i' and bounded by T_i', which carries a fundamental 2-cycle γ_i'.

Suppose that γ_i bounds in $V_0 \cap M_i$. Then $p_i(\gamma_i)$ bounds in T_0 and consequently so does $p_i(\gamma_i')$. But $g_i(\gamma_0)$ is a linear combination of γ_i and γ_i'. Hence $p_i g_i(\gamma_0)$ bounds in T_0 for sufficiently large i—*a* contradiction. Also, if a subsequence $\{T_{n_i}\}$ of $\{T_i\}$ converges to a proper subset of T_0, $p_{n_i}(\gamma_{n_i})$ bounds in T_0 and another contradiction arises.. Thus $\{T_i\}$ converges strongly to T_0.

Denote by U_i the interior of $N_i \cup U_i'$. If $T_i \cup U_i$ lies in V_0, then γ_i bounds in $V_0 \cap M_i$ and $p_i(\gamma_i)$ bounds in T_0 for sufficiently large i. This contradiction and the 2-regularity of the convergence of $\{M_i\}$ to M_0 imply that $\{T_i \cup U_i\}$ converges to $T_0 \cup U_0$ and $\{M_i - U_i\}$ converges to $M_0 - U_0$.

3. Regular mappings whose inverses are 3-cells. In this section, M_i denotes a 3-cell and K_i its boundary. The remainder of the notation is that of Section 2. The 3-cell M_0 may be assumed to consist of those points $(x_1, \cdots, x_n)$ of E^n for which $x_1^2 + x_2^2 + x_3^2 = 1$ and $x_i = 0$ for $4 \leqq i \leqq n$. It follows from results in Section 2 that are sequences $\{g_i\}$ and $\{g_i'\}$ of mappings and a sequence $\{\epsilon_i\}$ of positive numbers converging to 0 such that for each i, g_i and g_i' are piecewise linear ϵ_i-mappings M_0 onto M_i and M_i onto M_0 such that $g_i \mid K_0$ and $g_i \mid K_i$ are homeomorphisms and $g_i(M_0 - K_0) = M_i - K_i = g_i'^{-1}(M_0 - K_0)$.

Definition 3.1. If M is a 3-cell bounded by a 2-sphere K and t is an arc lying except for its endpoints, which lie in K, in $\operatorname{int} M$, then t is said to be *unknotted* in M provided that there is a piecewise linear homomorphism carrying M onto a (solid) cube S in E^3 and carrying t onto a straight line interval in S. If A is an annulus except for its boundary, which lies in K, in $\operatorname{int} M$, then A is said to be unknotted in M provided that there is a piecewise linear homeomorphism of M onto S which carries A onto the union of all intervals in S intersecting a triangle on $\operatorname{bdry} S$ and parallel to a fixed interval in S.

LEMMA 3.2. *Suppose that t_0 is an unknotted polyhedral arc in M_0 with endpoints P_0' and P_0'' such that $t_0 \cap K_0 = P_0' \cup P_0''$. Then there is a sequence $t_1, t_2, \cdots$ of arcs converging to t_0 such that* (1) *for each i, t_i is unknotted in M_i and has endpoints P_i' and P_i'' and* (2) *the sequences $\{P_i'\}$ and $\{P_i''\}$ converge to P_0' and P_0''.*

Proof. There is a disc D_0 in M_0 with boundary J_0 such that $t_0 \subset D_0$ and $D_0 \cap K_0 = J_0$. It follows from Lemma 2.13 and the corollary to Theorem 2.14 that there is a sequence $D_1, D_2, \cdots$ of discs converging to D_0 such that for each i, $D_i \subset M_i$, $D_i \cap K_i = \text{bdry}\, D_i = J_i$ and the sequences $\{J_i\}$ converges h-0-regularly to J_0. Let E_0 be a disc which is the closure of one of the components of $K_0 - J_0$ and let $E_1, E_2, \cdots$ be a sequence of discs converging to E_0 such that for each i, E_i is the closure of a component of $K_i - J_i$. Then the sequence of 2-spheres, $\{E_i \cup D_i\}$ converges strongly to $E_0 \cup D_0$. If s_0 is an arc in E_0 such that $s_0 \cap J_0 = P_0' \cup P_0''$, it follows from Lemma 2.5, applied to the natural mapping of a regular neighborhood of $E_0 \cup D_0$ onto $E_0 \cup D_0$ that there is a sequence $C_1, C_2, \cdots$ of simple closed curves converging strongly to $s_0 \cup t_0$ such that for each i, $C_i \subset D_i \cup E_i$. If ϵ is a positive number, then for sufficiently large i, there is an arc t_i in $C_i \cap D_i$ with endpoints P_i' and P_i'' such that $t_i \cap J_i = P_i' \cup P_i''$, $d(P_i', P_0') < \epsilon$ and $d(P_i'', P_0'') < \epsilon$. The sequence $\{t_i\}$ is the required sequence.

LEMMA 3.3. *Suppose that A_0 is an annulus in M_0 bounded by the simple closed curves J_0' and J_0'' such that $A_0 \cap K_0 = J_0' \cup J_0''$ and that U_0 and V_0 are the components of $M_0 - A_0$, the closure of U_0 being a 3-cell. Then there is a sequence of annuli, $\{A_i\}$, converging strongly to A_0 such that for each i, A_i lies in M_i and is bounded by simple closed curves J_i' and J_i'' such that $J_i' \cup J_i'' = A_i \cap K_i$, the sequences $\{J_i'\}$ and $\{J_i''\}$ converge h-0-regularly to J_0' and J_0'' and if U_i and V_i represent the components of $M_i - A_i$, the closure of U_i being a 3-cell, $\{U_i\}$ converges to $A_0 \cup U_0$ and $\{V_i\}$ converges to $A_0 \cup V_0$.*

Proof. It follows from the corollary to Theorem 2.14 that there are sequences $\{J_i'\}$ and $\{J_i''\}$ of simple closed curves converging h-0-regularly to J_0' and J_0'' such that for each i, J_i' and J_i'' lie in K_i. A slight modification of the proof of Lemma 2.13 implies the existence of a sequence $\{A_i'\}$ of singular annuli converging to A_0 such that for each i, A_i' lies in M_i and is bounded by $J_i' \cup J_i''$, $A_i' \cap K_i = J_i' \cup J_i''$, and there are no singularities on $J_i' \cup J_i''$. It follows from the generalization of Dehn's Lemma to annuli and other surface of genus 0 [15] that there is, arbitrarily close to A_i', a non-singular annulus A_i whose boundary is $J_i' \cup J_i''$ and which is such that

$A_i \cap K_i = J_i' \cup J_i''$. Each A_i may be so chosen that $\{A_i\}$ is the required sequence.

For each i, the set $K_i - (J_i' \cup J_i'')$ is the union of an open annulus B_i and two open discs D_i' and D_i'', $B_i \cup A_i$ is the boundary of the component V_i of $M_i - A_i$ and $D_i' \cup D_i'' \cup A_i$ is the boundary of the component U_i. Since $\{K_i\}$ converges 1-regularly to K_0, $\{A_i \cup B_i\}$ converges to $A_0 \cup B_0$ and $\{A_i \cup D_i' \cup D_i''\}$ converges to $A_0 \cup D_0' \cup D_0''$. The h-2-regular convergence of $\{M_i\}$ to M_0 implies that no point of $V_0 \cup B_0$ is a sequential limit point of any subsequence of $\{U_i\}$ and that no point of $U_0 - (D_0' \cup D_0'')$ is a sequential limit point of any subsequence of $\{V_i\}$. Thus $\{U_i\}$ and $\{V_i\}$ converge to $A_0 \cup U_0$ and $A_0 \cup V_0$.

COROLLARY. *If C_0 is a simple closed curve in A_0 separating J_0' from J_0'', then there is a sequence $\{C_i\}$ of simple closed curves converging strongly to C_0 such that for each i, $C_i \subset A_i$ and separates J_i' from J_i''.*

Proof. Since the sequence of 2-spheres, $\{A_i \cup D_i' \cup D_i''\}$ converges strongly to $A_0 \cup D_0' \cup D_0''$, the existence of a sequence $\{C_i\}$ converging strongly to C_0 such that for each i, $C_i \subset A_i$ is a direct consequence of an application of Lemma 2.5 to the natural mapping of a regular neighborhood in E^n of $A_0 \cup D_0' \cup D_0''$ onto $A_0 \cup D_0' \cup D''$. Suppose that U_0 is a regular neighborhood of C_0 consistently imbedded in W_0, a regular neighborhood of A_0, that p is the natural mapping, that p_i is a piecewise linear ϵ_i-approximation to $p \mid A_i$, and that γ_i is a non-trivial 1-cycle carried by C_i, which does not bound in $U_0 \cap M_i$. If C_i does not separate J_i' from J_i'' in A_i, then it bounds a disc in A_i. Thus γ_i bounds in A_i and $p_i(\gamma_i)$ bounds in A_0. Since $A_0 \cap U_0$ is a deformation retract of A_0, $p_i(\gamma_i)$ bounds in $A_0 \cap U_0$. Thus $g_i p_i(\gamma_i)$ bounds in $U_0 \cap M_i$. But, since for sufficiently large i and small ϵ_i, $g_i p_i(\gamma_i)$ is homologous to γ_i in $U_0 \cap M_i$, this is a contradiction.

LEMMA 3.4. *Suppose that A_0 is an annulus in M_0 whose boundary curves J_0' and J_0'' are such that $A_0 \cap K_0 = J_0' \cup J_0''$. Suppose, further, that B_0 is an annulus in $\operatorname{int} A_0$ whose boundary curves C_0' and C_0'' are such that C_0' separates J_0' from C_0'' which separates C_0' from J_0'' in A_0. Then there is a sequence $\{B_i\}$ of annuli converging strongly to B_0 such that for each i, $B_i \subset \operatorname{int} M_i$.*

Proof. Let $\{A_i\}$ be a sequence of annuli whose existence is implied by Lemma 3.3 such that (1) for each i, A_i is bounded by simple closed curves J_i' and J_i'' such that $A_i \cap K_i = J_i' \cup J_i''$, (2) the sequences $\{J_i'\}$ and $\{J_i''\}$ converge to J_0' and J_0'' h-0-regularly and (3) $\{A_i\}$ converges strongly to A_0.

Let ϵ be a positive number. Denote by Z_0' and Z_0'' simple closed curves in A_0 such that (1) Z_0' separates J_0' from C_0' and Z_0'' separates C_0'' from J_0'' and (2) the annuli in A_0, F_0' and F_0'', bounded by Z_0' and C_0' and C_0'' and Z_0'' respectively lie in $\epsilon/2$-neighborhoods of C_0' and C_0''. There are tori, T_0' and T_0'', which, together with their interiors, U_0' and U_0'', form regular $\epsilon/4$-neighborhoods of Z_0' and Z_0'', neither intersecting $B_0 \cup J_0' \cup J_0''$. From Lemma 2.16 it follows that there are sequences $\{T_i'\}$ and $\{T_i''\}$ of compact 2-manifolds converging strongly to T_0' and T_0'' such that for each i, T_i' and T_i'' are mutually exclusive subsets of $\operatorname{int} M_i$ with interiors U_i' and U_i'' and the sequences $\{U_i'\}$ and $\{U_i''\}$ converge to $U_0' \cup T_0'$ and $U_0'' \cup T_0''$. Denote by V_0' and V_0'' regular $\epsilon/4$-neighborhoods of Z_0' and Z_0'' in E^n such that $V_0'' \cap M_0$ is a regular neighborhood of $U_0'' \cup T_0''$ and $V_0' \cap M_0$ is a regular neighborhood of $U_0' \cup T_0'$ and by W_0 a regular $\epsilon/4$-neighborhood of A_0 in E^n with natural mapping p in which V_0' and V_0'' are consistently imbedded. For sufficiently large i, $U_i' \subset V_0'$ and $U_i'' \subset V_0''$.

It follows from the corollary to Lemma 3.3 that there are sequences $\{C_i'\}$ and $\{C_i''\}$ of simple closed curves converging strongly to C_0' and C_0'' such that for each i, each of C_i' and C_i'' lies in A_i and separates J_i' from J_i''. It is conceivable that C_i' fails to separate J_i' from C_i'' in A_i. If this is the case, C_i'' separates J_i' from C_i'. Denote by U_0 a regular $\epsilon/4$-neighborhood of C_0' not intersecting $J_0' \cup C_0''$, by T_0 its torus boundary, by $T_1, T_2, \cdots$ a sequence of 2-manifolds converging strongly to T_0 and by $U_1, U_2, \cdots$ the interiors in $M_1, M_2, \cdots$ of $T_1, T_2, \cdots$. The annuli $R_0, R_1, \cdots$ bounded by $J_0' \cup C_0'$, $J_1' \cup C_0''$, $\cdots$ converge to a subset of A_0 which contains R_0. Hence for sufficiently large i, R_i intersects U_i and consequently T_i. Small changes may be made in T_i so that each component of $R_i \cap T_i$ is a simple closed curve. For sufficiently large i, $R_i \cap T_i$ separates J_i' *from* C_i'' in R_i and consequently one of the simple closed curve components, C_i, of $R_i \cap T_i$ separates J_i' from C_i''. If a non-trivial 1-cycle, γ_i, carried by C_i bounds in W_0, then, since C_i'' is deformable into C_i in R_i, C_i'' carries a non-trivial cycle which bounds in W_0, which contradicts the strong convergence of $\{C_i''\}$ to C_0''. If this argument is applied to a sequence of regular neighborhoods, U_0, of C_0' converging to C_0', a sequence $C_1, C_2, \cdots$ of simple closed curves is found which converges strongly to C_0' and which is such that for each i, C_i lies in A_i and separates J_i' from C_i''. Then C_i' may be replaced by C_i. Thus it may be assumed that C_i' does separate J_i' from C_i''. Denote the annulus in A_i bounded by $C_i' \cup C_i''$ by B_i'.

Let H_0', G_0 and H_0'' denote the three components of $W_0 - (V_0' \cup V_0'')$, H_0' containing J_0' and H_0'' containing J_0''. For sufficiently large i, $B_i' \subset W_0$.

$T_i' \cap B_i'$ separates $B_i' \cap G_0$ from $B_i' \cap H_0'$ and $B_i' \cap T_0''$ separates $B_i' \cap G_0$ from $B_i' \cap H_0''$. Small changes may be made in T_0' and T_i'' so that each component of $B_i' \cap T_i'$ and $B_i' \cap T_i''$ is a simple closed curve. If t is such a component and is not contractible in W_0, t separates C_i' from C_i'' in B_0', for otherwise, t would bound a disc in W_0, which is impossible. Thus the components of $B_i' \cap (T_i' \cup T_i'')$ which are not contractible in W_0 may be arranged in a sequence α, each element of which separates the elements preceding it from the elements following it in B_i'. Two elements of α will be called *joinable* if they are subsets of the same one of T_i' and T_i''. If t and t' are joinable in, say, T_i', then, since $p(t)$ is deformable into $p(t')$ in A_0 and consequently in $V_0' \cap A_0$, it follows from an application of Lemma 2.6 that for sufficiently large i, t is deformable into t' in $V_0' \cap M_i$ (i.e. $g_i p(t)$ is deformable into $g_i p(t')$ in $V_0' \cap M_i$ and for sufficiently large i and small ϵ_i, t is deformable into $g_i p(t)$ and t' into $g_i p(t')$ in $V_0' \cap M_i$).

Let s_1 denote the first element of α and s_2 the last element of α joinable to s_1. Let s_3 denote the first element of α between s_2 and C_2'', if such exists, not joinable to s_2 (otherwise, let $s_3 = C_2''$) and s_4 the last such element. Denote by R_1, R_2, and R_3 the annuli in A_1 bounded by C_i' and s_1, s_2 and s_3, and s_4 and C_i''. If t is a component of $R_j \cap (T_i' \cup T_i'')$ other than s_1, s_2, s_3, or s_4, t is not in α, so bounds a singular disc D_t in $V_0' \cap M_i$ or $V_0'' \cap M_i$ and a non-singular disc in A_i. If t is not contained in any other such disc in A_i, replace the disc in A_i bounded by t by D_t for each such t and replace the annuli in A_i bounded by s_1 and s_2 and s_3 and s_4 by singular annuli with the same boundaries in $V_0' \cap M_i$ and $V_0'' \cap M_i$. In this way, B_i' is replaced by a singular annulus having no singularities on its boundary, $C_i' \cup C_i''$, and not intersecting $H_0' \cup H_0''$. Thus it follows from the extension of Dehn's Lemma that $C_i' \cup C_i''$ bounds a non-singular annulus B_i in $M_i \cap W_0$ which does not intersect $H_0' \cup H_0''$ and thus lies in an ϵ-neighborhood of B_0. Since ϵ was arbitrary, the existence of the required sequence is established.

COROLLARY. *The Lemma remains true if one of the curves C_0' and C_0'' is J_0' or J_0''.*

Note. The extension of Dehn's Lemma demonstrates that, since B_i' lies in the annulus A_i, B_i may be constructed so that $\operatorname{int} B_i \cap (A_i - B_i')$ lies outside some small neighborhood of $C_i' \cup C_i''$.

LEMMA 3.5. *Suppose that A_0 is an annulus in M_0 bounded by simple closed surves J_0' and J_0'' such that $A_0 \cap K_0 = J_0' \cup J_0''$. Suppose, further, that $J_0' = t_{00}, t_{01}, \cdots, t_{0m} = J_0''$ is a sequence of simple closed curves in A_0,*

in that order, each separating J_0' from J_0'', and that $A_{01}, A_{02}, \cdots, A_{0m}$ are the annuli in A_0 bounded by consecutive pairs of elements of $\{t_{0j}\}$. Then there is a sequence $\{A_i\}$ of annuli converging strongly to A_0 such that for each i, (1) A_i lies in M_i, is bounded by simple closed curves J_i' and J_i'' and $A_i \cap K_i = J_i' \cup J_i''$, (2) there is a sequence $J_i' = t_{i0}, \cdots, t_{im} = J_i''$ of simple closed curves in A_i in that order, A_{ij} denoting the annulus in A_i whose boundary is $t_{ij-1} \cup t_{ij}$ and (3) the elements of $\{t_{ij}\}_j$ may be so selected that $\{t_{ij}\}_i$ converges strongly to t_{0j} and $\{A_{ij}\}_i$ converges to A_{0j}.

Proof. Suppose that ϵ is a positive number, W_0 is a regular ϵ-neighborhood in E^n of A_0 and for each j, W_{0j} is a regular ϵ-neighborhood in E^n of A_{0j} consistently imbedded in W_0 such that $W_{0j-1} \cap W_{0j} = V_{0j-1}$ is a regular neighborhood of t_{0j-1}, $W_{0j} \cap W_{0k} = 0$ unless $|j-k| \leqq 1$, and $W_{0j} \cap K_0 = 0$ unless $j = 0, m$. If, for sufficiently large i, an annulus A_i may be found which satisfies conditions (1) and (2) of the statement of the lemma and which is such that $A_{ij} \subset W_{0j}$ and $t_{ij} \subset V_{0j}$ and is not contractible in V_{0j}, then the lemma is proved. (If t_{ij} carries a nontrivial cycle which bounds in V_{0j}, then $p_{ij}(t_{ij})$ carries a nontrivial cycle which bounds in $V_{0j} \cap A_0$, where p_{ij} ϵ_i-approximates the projection map of V_{0j} into $V_{0j} \cap A_0$. Thus $p_{ij}(t_{ij})$ is contractible in $V_{0j} \cap A_0$ and t_{0j} is contractible in V_{0j}.)

For each $j = 2, 3, \cdots, m$, let s_{0j} denote a simple closed curve in $A_{0j} \cap V_{0j-1}$ and let s_{01} be a simple closed curve in an ϵ-neighborhood of t_{00} such that for each j, s_{0j} separates t_{0j-1} from t_{0j} in A_0. Let R_{0j} and T_{0j} denote the annuli in A_{0j} bounded by t_{0j-1} and s_{0j} and s_{0j} and t_{0j}. Lemma 3.4, particularly the remarks in the second part of its proof, implies the existence, for sufficiently large i, of sequences of simple closed curves, $t_{i0}', t_{i1}', \cdots, t_{im}'$ and $s_{i1}', s_{i2}', \cdots, s_{im}'$ and sequences of mutually exclusive annuli $\{R_{ij}'\}_j$ and $\{T_{ij}'\}_j$ such that for each j, (1) $R_{ij}'$is bounded by $t_{ij-1}' \cup s_{ij}'$ and T_{ij}' is bounded by $s_{ij}' \cup t_{ij}'$, (2) $R_{ij}' \subset V_{ij-1}$, (3) $T_{ij}' \subset W_{0j}$, (4) s_{ij}' and t_{ij-1}' are not contractible in V_{0j-1}, and (5) $(R_{i1}' \cup T_{i1}') \cap K_i = t_{i0}'$ and $(R_{im}' \cup T_{im}') \cap K_i = t_{im}'$. Small changes may be made in these annuli so that each component of $(T_{ij-1}' \cup T_{ij}') \cap R_{ij}'$ is a simple closed curve. (See the note following the proof of Lemma 3.4.) Arrange the components of $T_{ij}' \cap (R_{ij}' \cup R_{ij+1}')$ which are not contractible in W_0 in a sequence α_{ij}, the order in the secquence being determined by the order of the components in T_{ij}' from s_{ij}' to t_{ij}'.

Denote t_{i0}' by t_{i0}, t_{im}' by t_{im}, the last element of α_{ij} in $R_{ij}' \cap T_{ij}'$ by s_{ij} and the first element of α_{ij} following s_{ij} by t_{ij}. The simple closed curve t_{ij} lies in R_{ij+1}'. Let R_{ij}'' denote the annulus in R_{ij}' bounded by t_{ij-1} and s_{ij} and T_{ij}'' the annulus in T_{ij}' bounded by s_{ij} and t_{ij}. Then $T_{ij}'' \cap T_{ik}'' = 0$ and

$R_{ij}'' \cap R_{ik}'' = 0$ unless $j = k$ and each component of $(R_{ij}'' \cup R_{ij+1}'') \cap T_{ij}''$ other than s_{ij} and t_{ij} is a simple closed curve which is contractible in W_{0j} and hence bounds a disc in R_{ij}'' or R_{ij+1}''.

Consider $H_{ij} = R_{ij}'' \cap (T_{ij-1}'' \cup T_{ij}'')$. If t is a simple closed curve component of H_{ij} whose interior, D, in R_{ij}'' does not intersect H_{ij} and t lies in, say, T_{ij-1}'', t bounds a disc F in T_{ij-1}''. Replace F by D and move the adjusted T_{ij-1}'' slightly away from R_{ij}'' in such a way that no new intersections with any T_{ij}'' or R_{ij}'' are added. The adjusted T_{ij-1}'' still lies in W_{0j-1} and has one less simple closed curve in common with R_{ij}''. If this process is repeated until all the components of H_{ij} are removed except t_{ij-1} and s_{ij}, T_{ij}'' and R_{ij}'' are replaced by annuli T_{ij} and R_{ij} such that (1) $T_{ij} \cap T_{ih} = 0$ unless $j = h$ and $R_{ij} \cap R_{ih} = 0$ unless $j = h$, (2) $T_{ij} \cap R_{ij} = s_{ij}$, (3) $T_{ij} \cap R_{ij+1} = t_{ij}$, (4) $R_{ij} \subset V_{0j}$, and (5) $T_{ij} \subset W_{0j}$. The set $R_{ij} \cup T_{ij}$ is thus an annulus A_{ij} in W_{0j} and the annulus $A_i = \cup_j A_{ij}$, together with the sequences $\{A_{ij}\}_j$ and $\{t_{ij}\}_j$, satisfies conditions (1) and (2) of the statement of the lemma and the conditions stated in the first paragraph of this proof. This implies the truth of Lemma 3. 5.

Lemma 3. 6. *Suppose that R is a 3-cell with boundary S, t is an unknotted (polyhedral) arc in R with endpoints P' and P'' such that $t \cap S = P' \cup P''$ and A is a (polyhedral) annulus in R with boundary curves J' and J'' such that* (1) *$A \cap S = J' \cup J''$,* (2) *the discs D' and D'' in S bounded by J' and J'' contain P' and P'' respectively and* (3) *t lies in the component of $R - A$ whose closure is a 3-cell. Then A is unknotted.*

Proof. There is a disc D in R with boundary J such that $D \cap S = J$, $J \cap J'$ and $J \cap J''$ each consists of just two points and $t \subset D$. Small changes may be made in D so that each component of $D \cap A$ is either an arc or a simple closed curve. If a component s of $D \cap A$ is an arc, the endpoints of s lie in $J' \cup J''$. If both endpoints lie in, say, J', then $s \cup (D' \cap J)$ is a simple closed curve in D, so s intersects t—a contradiction. Hence one endpoint of s lies in J' and the other in J''. There is only one other arc as a component of $D \cap A$. Call it u. If a simple closed curve, w, is a component of $D \cap A$, it bounds a disc, F, in A, for if this is false, w and J' bound an annulus, K, in A and w bounds a disc, E, in D which does not intersect t. Thus $E \cup K$ is a singular disc in $R - t$ bounded by J'. This, however, is impossible. If $F - w$ does not intersect $D \cap A$, replace E in D by F and move it slightly away from A so that the adjusted D has one less simple closed curve in common with A. Repeat this process until a disc D' is obtained which contains t, is bounded by J and is such that $D \cap A = s \cup u$. That A is unknotted now follows readily.

THEOREM 3.7. *If t_0 is an unknotted arc in M_0 with endpoints P_0 and Q_0 such that $t_0 \cap K_0 = P_0 \cup Q_0$, then there is a sequence of arcs $\{t_i\}$ with endpoints $\{P_i \cup Q_i\}$ converging h-0-regularly to t_0 such that for each i, (1) t_i lies in M_i and is unknotted in M_i, (2) $t_i \cap K_i = P_i \cup Q_i$ and (3) the sequences $\{P_i\}$ and $\{Q_i\}$ converge to P_0 and Q_0.*

Proof. Let $P_0 = P_{00}, P_{01}, \cdots, P_{0m} = Q_0$ be a sequence of points of t_0 in that order, let W_0 be a regular neighborhood in E^n of t_0 and for each j, let W_{0j} be a regular neighborhood of the subarc $P_{0j-1}P_{0j}$ of t_0 which is consistently imbedded in W_0 and is such that $W_{0j} \cap W_{0j+1}$ is a regular neighborhood of P_{0j}, $W_{0j} \cap W_{0k} = 0$ unless $|j-k| \leqq 1$, and $W_{0j} \cap K_0 = 0$ unless $j = 0, m$. The theorem will be proved if it can be shown that for sufficiently large i there is an arc t_i satisfying conditions (1) and (2) which lies in W_0 and is such that for each j, $t_i \cap (W_{0j} \cup W_{0j+1})$ has only one component which intersects both W_{0j-1} and W_{0j+2}.

There is an unknotted annulus A_0 in $W_0 \cap M_0$ with boundary $t_{00} \cup t_{0m}$ such that (1) $A_0 \cap K_0 = t_{00} \cup t_{0m}$, (2) t lies in the component of $M_0 - A_0$ whose closure is a 3-cell and (3) there are simple closed curves $t_{00}, t_{01}, \cdots, t_{0m}$ in that order in A_0 such that for each j, $t_{0j-1} \cup t_{0j}$ bounds an annulus A_{0j} in $A_0 \cap W_{0j}$. From Lemma 3.2 it follows that there is a sequence $s_1, s_2, \cdots$ of arcs converging to t_0 such that for each i, s_i is unknotted in M_i, s_i has endpoints P_i' and Q_i', $s_i \cap K_i = P_i' \cup Q_i'$ and the sequences $\{P_i'\}$ and $\{Q_i'\}$ converge to P_0 and Q_0. It follows from Lemmas 3.3 and 3.5 that for sufficiently large i there is an annulus A_i in $W_0 \cap M_i$ bounded by simple closed curves t_{i0} and t_{im} such that (1) $A_i \cap K_i = t_{i0} \cup t_{im}$, (2) s_i lies in the component of $M_i - A_i$ whose closure is a 3-cell and (3) there are simple closed curves $t_{i0}, \cdots, t_{im}$ in that order in A_i such that for each j, $t_{ij-1} \cup t_{ij}$ bounds an annulus A_{ij} in $A_i \cap W_{0j}$. Let t_i denote an arc in A_i with endpoints P_i and Q_i such that (1) $t_i \cap t_{i0} = P_i$, (2) $t_i \cap t_{im} = Q_i$, (3) for each j, $t_i \cap A_{ij}$ is an arc. Since s_i is unknotted, it follows from Lemma 3.6 that A_i and therefore t_i is unknotted. Clearly there are not two components of $t_i \cap (W_{0j} \cup W_{0j+1})$ which intersect both W_{0j-1} and W_{0j+2} so that t_i is the required arc.

LEMMA 3.8. *Let t_0' and t_0'' be mutually exclusive unknotted arcs in M_0 such that $t_0' \cap K_0$ and $t_0'' \cap K_0$ are the unions of the endpoints of t_0' and t_0'' and let D_0 be a disc in M_0 whose boundary, J_0, contains t_0' and t_0'' and is such that* $\mathrm{cl}(J_0 - (t_0' \cup t_0'')) = D_0 \cap K_0$. *Then if $\{t_i'\}$ and $\{t_i''\}$ are sequences of arcs converging regularly to t_0' and t_0'' such that for each i, t_i' and t_i'' are unknotted in M_i and meet K_i only in their endpoints, there exists*

a sequence of discs $D_1, D_2, \cdots$ *converging to* D_0 *whose boundaries* $J_1, J_2, \cdots$ *converge regularly to* J_0 *such that for each* i, $t_i' \cup t_i'' \subset J_i$ *and*

$$\mathrm{cl}(J_i - (t_i' \cup t_i'')) = D_i \cap K_i.$$

Proof. There is a disc E_0 in M_0 which contains D_0 and whose boundary, B_0 is such that $E_0 \cap K_0 = B_0$. Also, there is a disc F_0 in M_0 with boundary C_0 such that $F_0 \cap E_0$ is an arc t_0 which separates t_0' from t_0'' in E_0, $F_0 \cap K_0 = C_0$, and $C_0 \cap B_0$ is the union of the endpoints of t_0. From Lemma 2.13 it follows that there is a sequence of discs, $F_1, F_2, \cdots$ converging to F_0 whose boundaries $C_1, C_2, \cdots$ converge regularly to C_0 and are such that for each i, $F_i \cap K_i = C_i$. If, for each i, U_i' and U_i'' denote the closures of the components of $M_i - F_i$, where $t_0' \subset U_0'$ and $t_0'' \subset U_0''$, the h-2-regularity of the convergence of $\{M_i\}$ to M_0 implies that the sequences $\{U_i'\}$ and $\{U_i''\}$ converge to U_0' and U_0''. Thus, for sufficiently large i, $t_i' \subset U_i'$ and $t_i'' \subset U_i''$. It is clear, then, that there is a disc E_i''' in M_i containing $t_i' \cup t_i''$ whose boundary, B_i, is such that $E_i''' \cap K_i = B_i$ and the sequence $\{B_i\}$ may be so selected that it converges regularly to B_0.

The E_i''' must be adjusted in such a way that the resulting sequence converges to E_0. Let V_0 be a regular ϵ-neighborhood of E_0 in M_0 whose boundary consists of an annulus in K_0 and the discs E_0' and E_0'' whose boundaries, B_0' and B_0'' are such that $E_0' \cap K_0 = B_0'$ and $E_0'' \cap K_0 = B_0''$. There are sequences $\{E_i'\}$ and $\{E_i''\}$ of discs converging strongly to E_0' and E_0'' such that for each i, $E_i' \cup E_i'' \subset M_i$, $E_i' \cap K_i = \mathrm{bdry}\, E_i'$ and $E_i'' \cap K_i = \mathrm{bdry}\, E_i''$. Denote by V_i the closure of the component of $M_i - (E_i' \cup E_i'')$ which contains $t_i' \cup t_i''$. Clearly $B_i \subset V_i$ and $\{V_i\}$ converges to V_0. The E_i' and E_i'' may be adjusted so that each component of $(E_i' \cup E_i'') \cap E_i'''$ is a simple closed curve. A push-pull argument proves that each E_i''' may be replaced by a disc E_i which contains $t_i' \cup t_i''$, lies in V_i, is bounded by B_i and meets K_i only in B_i. Since ϵ was arbitrary, the existence of a sequence of discs $\{E_i\}$ converging to E_0 such that for each i, $E_i \subset M_i$, $E_i \cap K_i = B_i$ and $t_i' \cup t_i'' \subset E_i$ is established. The proof of Lemma 3.4 may now be applied, with the obvious changes, to construct the required sequence $\{D_i\}$.

Note. In the application of Dehn's Lemma here, D_i may be so constructed that $\mathrm{int}\, D_i \cap (E_i - D_i')$, where D_i' is the disc in E_i replaced by D_i, does not intersect some small neighborhood of $t_i' \cup t_i''$. Also, the above proof demonstrates that if G_0 is the closure of the component of $E_0 - D_0$ whose boundary contains t_0' and $G_1, G_2, \cdots$ is a sequence of discs converging to G_0 such that for each i, $G_i \subset M_i$ and $\mathrm{bdry}\, G_i = t_i' \cup (G_i \cap K_i)$, then E_i may be

so constructed that $G_i \subset E_i$. Thus $(\operatorname{int} D_i) \cap G_i$ has no points in some small neighborhood of t_i'.

LEMMA 3.9. *Suppose that D_0 is a disc in M_0 with boundary J_0 such that $D_0 \cap K_0 = J_0$. Suppose, further, that $t_{00}, \cdots, t_{0m}$ is a sequence of arcs in D_0 such that (1) $t_{00} \cup t_{0m} \subset J_0$, (2) $t_{0j} \cap J_0$ is, for $j \neq 0, m$, the union of the endpoints of t_{0j} and (3) t_{0j} separates t_{0j-1} from t_{0j+1} in D_0. Denote by D_0 the disc in D_{0j} which is the closure of the component of $D_0 - \cup t_{0j}$ whose boundary contains $t_{0j-1} \cup t_{0j}$. Then there is a sequence $\{D_i\}$ of discs converging to D_0 whose boundaries $\{J_i\}$ converge regularly to J_0 such that for each i, (1) $D_i \cap K_i = J_i$ and (2) there are arcs $t_{i0}, t_{i1}, \cdots, t_{im}$ in D_i such that $t_{i0} \cup t_{im} \subset J_i$, $t_{ij} \cap J_i$ is the union of the endpoints of t_{ij} for $j \neq 0, m$ and $t_{ij-1} \cup t_{ij}$ lies on the boundary of a disc D_{ij}, which is the closure of a component of $D_i - \cup{}_j t_{ij}$. Furthermore, the sequence $\{t_{ij}\}_i$ converges h-0-regularly to t_{0j} and $\{D_{ij}\}_i$ converges to D_{0j}.*

Proof. Let W_0 be a regular neighborhood of D_0 in E^n and for each j, let W_{0j} be a regular neighborhood of D_{0j} consistently imbedded in W_0 such that $W_{0j} \cap W_{0j+1} = V_{0j}$ is a regular neighborhood of t_{0j} consistently imbedded in W_0, $W_{0j} \cap W_{0k} = 0$ unless $|j - k| \leqq 1$, and $W_{0j} \cap K_0 = 0$ unless $j = 0, m$. Let V_{00} and V_{0m} be regular neighborhoods of t_{00} and t_{0m} consistently imbedded in W_0 and meeting no other V_{0j}. If for each positive number ϵ it can be shown that for sufficiently large i, there is a disc D_i satisfying conditions (1) and (2) such that for each j, $D_{ij} \subset W_{ij}$ and t_{ij} is ϵ-homeomorphic to t_{0j}, then the lemma is proved.

For each j, let s_{0j} be an arc in $D_{0j} \cap V_{0j-1}$ which lies except for its endpoints in $\operatorname{int} D_{0j}$ and separates t_{0j-1} from t_{0j} and let R_{0j} and T_{0j} be the discs into which s_{0j} separates D_{0j}, R_{0j} containing t_{0j-1} and T_{0j} containing t_{0j}. Construct s_{0j} so that R_{0j} lies in V_{0j-1}. Theorem 3.7 and Lemma 3.8 imply that for sufficiently large i, there are, for each j, (1) a simple closed curve J_i in K_i which is ϵ-homeomorphic to J_0, (2) arcs t_{ij-1} and s_{ij} in V_{0j-1} which are ϵ-homeomorphic to t_{0j-1} and s_{0j}, t_{i0} lying in J_i, t_{ij-1} lying except for its endpoints in $\operatorname{int} M_i$ if $j > 1$, (3) an arc t_{im} in J_i which is ϵ-homeomorphic to t_{0m}, (4) a disc R_{ij} which lies in V_{0j-1}, is bounded by $t_{ij-1} \cup s_{ij}$ and portions of J_i and meets K_i only in these portions of J_i and (5) a disc T_{ij}' which lies in W_{0j}, is bounded by $s_{ij} \cup t_{ij}$ and portions of J_i and meets K_i only in these portions of J_i. Clearly these arcs and discs may be so chosen that no two arcs intersect and $T_{ij}' \cap T_{ik}' = 0$ unless $j = k$.

Small changes in the R_{ij} may be made so that each component of $H_{ij} = (T_{ij-1}' \cup T_{ij}') \cap R_{ij}$ except t_{ij-1} and s_{ij} is a simple closed curve. (See

the note following Lemma 3.8.) If t is such a component in, say, T_{ij-1}', whose interior, E, in R_{ij} does not intersect H_{ij}, replace its interior in T_{ij-1}' by E and move it slightly away from R_{ij} so that H_{ij} has one less component and the adjusted T_{ij-1}', which still lies in W_{0j-1}, has no new intersection with any R_{ij} or T_{ij}'. Repeat this process until each T_{ij}' is replaced by a disc T_{ij} such that $T_{ij} \cap R_{ij} = s_{ij}$, $T_{ij-1} \cap R_{ij} = t_{ij-1}$ and $\cup (T_{ij} \cup R_{ij})$ is a disc D_i such that $D_i \cap K_i = J_i$. Denote $R_{ij} \cup T_{ij}$ by D_{ij}. The disc D_{ij} lies in W_{ij}, t_{ij} is ϵ-homeomorphic to t_{0j} and D_i satisfies conditions (1) and (2) of the statement of the lemma. Thus the existence of the required sequence is proved.

THEOREM 3.10. *If D_0 is a disc in M_0 with boundary J_0 such that $D_0 \cap K_0 = J_0$, then there is a sequence of discs $\{D_i\}$ with boundaries $\{J_i\}$ converging h-0-regularly to D_0 such that for each i, $D_i \subset M_i$ and $D_i \cap K_i = J_i$.*

Proof. Let $t_{00}, t_{01}, \cdots, t_{0m}$ be a sequence of arcs in D_0 such that $t_{00} \cup t_{0m} \subset J_0$, $t_{0j} \cap J_0$, for $j \neq 0, m$, is the union of the endpoint of t_{0j}, t_{0j} separates t_{0j-1} from t_{0j+1} in D_0 and $t_{0j-1} \cup t_{0j}$ is on the boundary of a disc E_{ij} which is the closure of a component of $D_0 - \cup t_{0j}$. Let $s_{00}, s_{01}, \cdots, s_{0m}$ be a sequence of arcs in D_0 such that $s_{00} \cup s_{0m} \subset J_0$, $s_{0j} \cap J_0$, for $j \neq 0, m$, is the union of the endpoints of s_{0j}, s_{0j} separates s_{0j-1} from s_{0j+1} in D_0 and $s_{0j} \cap t_{0k}$ is a point, P_{0jk}. Denote by D_{0j} the disc which is the closure of the component of $D_0 - \cup s_{0j}$ bounded in part by $s_{0j-1} \cup s_{0j}$, and by F_{0jk} the disc $D_{0j} \cap E_{0k}$. Let W be a regular neighborhood of D_0 in E^n and U_j a regular neighborhood of D_{0j} consistently imbedded in W such that $U_j \cap U_{j+1}$ is a regular neighborhood of s_{0j} consistently imbedded in W and $U_j \cup K_0 = 0$ unless $j = 0, m$. Suppose that $U_j \cap U_{j'} = 0$ unless $|j - j'| \leqq 1$. Also, let V_k be a regular neighborhood of E_{0k} consistently imbedded in W such that $V_k \cap V_{k+1}$ is a regular neighborhood of t_{0k} consistently imbedded in W, $V_k \cap V_{k'} = 0$ unless $|k - k'| \leqq 1$, $V_{0k} \cap U_{0j}$ is a regular neighborhood W_{jk} of F_{0jk} consistently imbedded in W, and $V_k \cap K_0 = 0$ unless $k = 0, m$.

It follows from the Lemma 3.9 that there is a sequence $\{D_i'\}$ of discs converging to D_0 whose boundaries, $\{J_i\}$ converge 0-regularly to J_0 such that for each i, (1) $D_i' \cap K_i = J_i$ and (2) there are arcs $t_{i0}, t_{i1}, \cdots, t_{im}$ in D_i' such that $t_{i0} \cup t_{im} \subset J_i$, $t_{ik} \cap J_i$ is, for $k \neq 0, m$, the union of the endpoints of t_{ik}, and $t_{ik-1} \cup t_{ik}$ lies in the boundary of a disc, E_{ik}', which is the closure of a component of $D_i' - \cup_k t_{ik}$ and lies, for sufficiently large i, in V_k. Furthermore, for each k, the sequence $\{t_{ik}\}_i$ converges regularly to t_{0k} and $\{E_{ik}'\}$ converges to E_{0k}. An application of the proof of Lemma 3.2 demonstrates the existence, for sufficiently large i and each j, of an arc s_{ij} in $U_j \cap U_{j+1} \cap D_i$ such that $s_{i0} \cup s_{im} \subset J_i$, $s_{ij} \cap J_i$ is the union of the endpoints of s_{ij} for $j \neq 0, m$

and $s_{ij} \cap t_{ik}$ is a point, P_{ijk}. Denote by D_{ij}' the closure of the component of $D_i' - \cup_j s_{ij}$ whose boundary contains $s_{ij-1} \cup s_{ij}$ and by F_{ijk}' the disc $D_{ij}' \cap E_{ik}'$.

It may be assumed that each W_{jk} has diameter less than $\epsilon/100$. The theorem will be proved if it can be shown that there is a positive number δ such that for sufficiently large i, there is a disc D_i in $W \cap M_i$ such that $D_i \cap K_i = J_i$ and each pair of points in D_i whose distance apart is less than δ bounds an arc in D_i of diameter less than ϵ. Let δ be such that if p and q are points in W whose distance apart is less than δ, then for some j and k,

$$p \cup q \subset W_{jk} - W_{jk} \cap (V_{k'} \cup U_{j'}),$$

where $|k - k'| = |j - j'| = 1$. The remainder of the proof is devoted to the demonstration of the existence for sufficiently large i of a disc D_i satisfying the above conditions with respect to δ.

For each k, let V_k' denote a regular neighborhood of E_{0k} such that $V_k' \subset \operatorname{int} V_k$. Denote $V_k' \cap U_j$ by W_{jk}'. It may be assumed that for sufficiently large i, $E_{ij}' \subset V_j'$. Denote by A_{0j} an annulus in $U_j \cap U_{j+1} \cap M_0$ such that (1) $A_{0j} \cap K_0 = \operatorname{bdry} A_{0j}$, (2) s_{0j} lies in the component of $M_0 - A_{0j}$ whose closure is a 3-cell and (3) $A_{0j} \cap W_{jk}'$ and $A_{0j} \cap W_{j+1k}'$ are annuli. It follows from Lemma 3. 5 that for sufficiently large i there is an annulus A_{ij} in $U_j \cap U_{j+1} \cap M_i$ such that (1) $A_{ij} \cap K_i = \operatorname{bdry} A_{ij}$, (2) s_{ij} lies in the component of $M_i - A_{ij}$ whose closure is a 3-cell, (3) each simple closed curve in $A_{ij} \cap W_{jk}'$ which is contractible in A_{ij} bounds a disc in $A_{ij} \cap W_{jk}$, and (4) $A_{ij} \cap D_i'$ separates $D_i' \cap \cup U_{j'}$, $j' < j$, from $D_i' \cap \cup U_{j'}$, $j' > j + 1$. Small adjustments may be made in A_{ij} so that each component of $A_{ij} \cap D_{ij}'$, except for an arc in D_{ij}' from t_{00} to t_{0m} and one in D_{ij+1}', is a simple closed curve which is contractible in A_{ij}. If $j' \neq j$, $j + 1$, each component of $D_{ij'}' \cap A_{ij}$ is, for some k, a subset of W_{jk}'.

Suppose that t is a component of $A_{ij} \cap D_i'$ which for some k lies in W_{jk}' and suppose, further, that the interior, E, of t in A_{ij} does not intersect D_i'. The interior, F, of t in D_i' lies either in $F_{ij'k'}'$, for some $j' \neq j$, $j+1$, $|k - k'| \leqq 1$, or in $F_{ijk-1}' \cup F_{ijk}' \cup F_{ijk+1}'$ or in $F_{ij+1k-1}' \cup F_{ij+1k}' \cup F_{ij+1k+1}'$ and may be replaced by E and then moved slightly away from A_{ij} so that the number of components of $D_i' \cap A_{ij}$ is reduced. If $\operatorname{int} E$ does intersect D_i', consider a component t' of $E \cap D_i'$ whise interior, E', does not intersect D_i'. Let F' denote the interior of t' in D_i'. The set E' lies in W_{jk} and F' can be replaced by E' as above. If this process is repeated, each $F_{ij'k}$, $j' \neq j$, $j+1$, is replaced by a disc $F_{ij'k}''$ whose boundary is that of $F_{ij'k}'$, which does not intersect both components of $W - (U_j \cap U_{j+1})$—i. e. does not intersect A_{ij}—and lies in V_k. Also, certain discs in D_{ij}' and D_{ij+1}' whose boundaries lie in

some W_{jk}' (or, in some cases, in W_{jk}) are replaced by discs in W_{jk}. Furthermore, $\cup F_{ijk}''$ is a disc, no s_{ij} is changed and no new intersections with any A_{ij} are added. This process is applied to each A_{ij}, starting with A_{i0}. Consider F_{ijk}'. After this process has been applied to each $A_{ij'}$, $j' < j-1$, F_{ijk}' has been replaced by a disc F_{jjk}'' whose boundary is that of F_{ijk}', which lies in V_k and $\cup U_{j'}$, $j' \geqq j-1$, and whose intersection with $A_{ij'}$, $j' \geqq j-1$, still lies in $W_{j'k}'$. After this process is applied to A_{ij-1} and A_{ij}, F_{ijk}'' is replaced by a disc F_{ijk} which lies in $W_{jk-1} \cup W_{jk} \cup W_{jk+1}$. The disc F_{ijk} is not affected by the action on $A_{ij'}$ for $j' > j$. Also, the resulting $\cup F_{ijk}$ is a disc D_i whose boundary is J_i.

Suppose that p and q are points of D_i whose distance apart is les than δ and that $p \cup q \subset W_{jk} - W_{jk} \cap (V_{k-1} \cup U_{j-1})$. Then for sufficiently large i, $p \cup q$ is a subset of the union of F_{ijk}, F_{ijk+1}, F_{ijk+2}, F_{ijk-1}, F_{ij+1k}, $F_{ij+1k+1}$, $F_{ij+1k+2}$, and $F_{ij+1k-1}$ which, since each W_{jk} has diameter less than $\epsilon/100$, certainly has diameter less than ϵ. This completes the proof of the theorem.

LEMMA 3.11. *If R_0 is a 3-cell in M_0 bounded by the 2-sphere S_0 and $S_1, S_2, \cdots$ is a sequence of 2-spheres converging h-0-regularly to S_0 such that for each i, S_i bounds the 3-cell R_i in M_i, then the sequence $\{R_i\}$ converges h-2-regularly to R_0.*

Proof. The h-2-regularity of the convergence of $\{M_i\}$ to M_0 implies that the convergence of $\{R_i\}$ to R_0 is regular at each point of $\operatorname{int} R_0$. Suppose that P is a point of S_0 and ϵ is a positive number. There is a positive number $\delta' < \epsilon/2$ such that each singular 2-sphere in M_i of diameter less than $2\delta'$ bounds a singular 3-cell in M_i of diameter less than $\epsilon/2$ and there is a positive number $\delta < \delta'/2$ such that each singular 1-sphere in S_i of diameter less than δ bounds a singular 2-cell in S_i of diameter less than $\delta'/2$.

Suppose that J_i is a singular j-sphere, $j \leqq 2$, which may be assumed to be polyhedral, in $R_i \cap S(P, \delta/4)$. It may be assumed that $J \cap S_i = 0$. Denote by S_0' and R_0' the boundary and closure respectively of the spherical neighborhood $M_0 \cap S(P, \delta/2)$. There is a sequence $\{S_i'\}$ of 2-spheres converging strongly to S_0' such that for each i, S_i' bounds a 3-cell, R_i', in M_i. The sequences $\{R_i'\}$ and $\{M_i - R_i'\}$ converge to R_0' and $\operatorname{cl}(M_0 - R_0')$. Hence, for sufficiently large i, $J_i \subset \operatorname{int} R_i'$. Adjust each S_i' slightly so that each component, K, of $S_i' \cap R_i$ is a subset of a disc which is bounded by a finite number of simple closed curves, each of which bounds a (non-singular) disc in S_i of diameter less than $\delta'/2$. Add these discs to K, moving them slightly away from S_i in such a way that K becomes a non-singular 2-sphere, K', in $S(P, \delta') \cap R_i$. At least one such 2-sphere, call it K_i, has J_i in its interior.

Since K_i has diameter less than $2\delta'$, it bounds a 3-cell C_i in M_i, consequently in R_i, of diameter less than $\epsilon/2$. But C_i contains J_i and therefore J_i bounds a $(j+1)$-cell in C_i which is a subset of $R_i \cap S(P,\epsilon)$. This proves the lemma.

THEOREM 3.12. *The sequence $M_1, M_2, \cdots$ converges to M_0 completely regularly.*

Proof. Suppose that ϵ is a positive number. It will be shown that for sufficiently large i, there is an ϵ-homeomorphism of M_0 onto M_i. The proof will consist of the construction for each i of a certain subdivision of M_i similar to that obtained for a geometric cube in E^3 by sections of planes parallel to its faces.

Denote by $D_{00}, D_{01}, \cdots, D_{0m}, E_{00}, E_{01}, \cdots, E_{0m}, F_{00}, F_{01}, \cdots, F_{0m}$ three sequences of mutually exclusive discs in M_0 such that (1) $D_{00} \cup D_{0m} \cup E_{00} \cup E_{0m} \cup F_{00} \cup F_{0m} \subset K_0$, (2) for each $p \neq 0$, (i) $D_{0p} \cap K_0 = \operatorname{bdry} D_{0p}$, $E_{0p} \cap K_0 = \operatorname{bdry} E_{0p}$ and $F_{0p} \cap K_0 = \operatorname{bdry} F_{0p}$ and (ii) D_{0p} separates D_{0p-1} from D_{0p+1}, E_{0p} separates E_{0p-1} from E_{0p+1} and F_{0p} separates F_{0p-1} from F_{0p+1} in M_0, (3) for each p, q, r, $D_{0p} \cap E_{0q}$, $E_{0q} \cap F_{0r}$, and $D_{0p} \cap F_{0r}$ are arcs and $D_{0p} \cap E_{0q} \cap F_{0r}$ is a point and (4) each component of $M - \bigcup (D_{0p} \cup E_{0p} \cup F_{0p})$ has diameter less than $\epsilon/2$.

It follows from Theorem 3.10 that there is, for each p, a sequence $\{D_{ip}\}$ of discs converging h-0-regularly to D_{0p} such that for each i, (1) D_{ip} is in M_i and $D_{ip} \cap K_i = \operatorname{bdry} D_{ip}$, $(p \neq 0, m)$, (2) $D_{i0} \cup D_{im} \subset K_i$, (3) D_{ip} separates D_{ip-1} from D_{ip+1} in M_i for $p \neq 0, m$. Denote by R_{ip} the closure of the component of $M_i - \bigcup_p D_{ip}$ whose boundary contains $D_{ip-1} \cup D_{ip}$. It follows from Lemma 3.11 that for each p, the sequence of 3-cells, $\{R_{ip}\}_i$ converges h-2-regularly to R_{0p} and that the lemmas and theorems already proved can be applied to these sequences.

If Theorem 3.10 is applied to each such sequence, it is shown that for each q there is a sequence $\{E_{iq}\}$ of discs converging h-0-regularly to E_{0q} such that (a) for each i, (1) E_{iq} is in M_i and $E_{iq} \cap K_i = \operatorname{bdry} E_{iq}$ for $q \neq 0, m$, (2) $E_{i0} \cup E_{im} \subset K_i$, (3) E_{iq} separates E_{iq-1} from E_{iq+1} in M_i for $q \neq 0, m$, and (4) $E_{iq} \cap D_{ip}$ is an arc for each p and (b) each sequence $\{E_{iq} \cap D_{ip}\}_i$ converges 0-regularly to $E_{0q} \cap D_{0p}$. Denote by R_{ipq} the closure of the component of $M_i - \bigcup (D_{ip} \cup E_{ip})$ whose boundary contains discs in E_{iq-1}, E_{iq}, D_{ip-1} and D_{ip}. The sequence $\{R_{ipq}\}_i$ converges h-2-regularly to R_{opq}.

If Theorem 3.10 is now applied to each sequence $\{R_{ipq}\}_i$, it is shown that for each r, there is a sequence $\{F_{ir}\}$ of discs converging h-0-regularly to F_{0r} such that (a) for each i, (1) F_{ir} is in M_i and $F_{ir} \cap K_i = \operatorname{bdry} F_{ir}$ for $r \neq 0, m$, (2) $F_{i0} \cup F_{im} \subset K_i$, (3) F_{ir} separates F_{ir-1} from F_{ir+1} in M_i for $r \neq 0, m$

and (4) for each p, q, $E_{iq} \cap F_{ir}$ and $D_{ip} \cap E_{iq} \cap F_{ir}$ is a point and (b) each sequence $\{E_{iq} \cap F_{ir}\}$ and $\{D_{ip} \cap F_{ir}\}_i$ converges h-0-regularly to $E_{oq} \cap F_{or}$ and $D_{op} \cap F_{or}$. Denote by R_{ipqr} the closure of that component of

$$M_i - \cup (D_{ip} \cup E_{ip} \cup F_{ip})$$

whose boundary contains discs in D_{ip-1}, D_{ip}, E_{iq-1}, E_{iq}, F_{ir-1}, and F_{ir}. The sequence $\{R_{ipqr}\}_i$ converges h-2-regularly to R_{opqr}.

Since every sequence of discs which converges 0-regularly to a disc converges completely regularly [6], it follows that for sufficiently large i, there is an $\epsilon/2$-homeomorphism h_i of $\cup (D_{0p} \cup E_{0p} \cup F_{0p})$ onto $\cup (D_{ip} \cup E_{ip} \cup F_{ip})$. For sufficiently large i, it is also true that R_{ipqr} lies in an $\epsilon/2$-neighborhood of R_{opqr} and has diameter less than $\epsilon/2$. A homeomorphism g_i of M_0 onto M_i which, for each p, q, r, extends $h_i \mid \text{bdry}\, R_{opqr}$ to a homeomorphism of R_{opqr} onto R_{ipqr} is an ϵ-homeomorphism. Thus Theorem 3.12 is proved.

A direct consequence of Theorem 3.12 is the main theorem of this section.

THEOREM 3.13. *If f is a homotopy 2-regular mapping of a metric space X onto a metric space Y such that each inverse under f is a 3-cell, then f is completely regular.*

4. Regular mappings whose inverses are 3-manifolds.

In this section, the notation of *section* 2 is used. Each M_i is a compact 3-manifold with boundary imbeddable in E^3 and the sequence $\{K_i\}$ converges completely regularly to K_0. Let $\{\epsilon_i\}$ be a sequence of positive numbers converging to 0 and $\{g_i\}$ be a sequence of mappings such that for each i, g_i is a piecewise linear ϵ_i homeomorphism of K_0 onto K_i.

LEMMA 4.1. *If D_0 is a disc in M_0 with boundary J_0 such that $D_0 \cap K_0 = J_0$, then there is a sequence $\{D_i\}$ of discs converging completely regularly to D_0 whose boundaries $\{J_i\}$ are such that for each i, $D_i \subset M_i$ and $D_i \cap K_i = J_i = g_i(J_0)$.*

Proof. If ϵ is a positive number, there is a regular ϵ-neighborhood A_0 of J_0 in K_0 which is an annulus bounded by simple closed curves J_0' and J_0''. Also, there are discs D_0' and D_0'' in M_0 such that $D_0' \cap K_0 = \text{bdry}\, D_0' = J_0'$, $D_0'' \cap K_0 = \text{bdry}\, D_0'' = J_0''$, and the 2-sphere $D_0' \cup A_0 \cup D_0''$ bounds a 3-cell C_0 in M_0 which is a regular ϵ-neighborhood of D_0. It follows from Lemma 2.13 that there are sequences of 2-cells, $\{D_i'\}$ and $\{D_i''\}$ converging strongly to D_0' and D_0'' such that for each i, $D_i' \cup D_i'' \subset M_i$, $D_i' \cap K_i = \text{bdry}\, D_i' = g_i(J_0')$ and $D_0'' \cap K_i = \text{bdry}\, D_0'' = g_i(J_0'')$. It follows from Lemma 2.15

that for sufficiently large i, the 2-sphere $D_i' \cup g_i(A_0) \cup D_i''$ bounds a 3-cell C_i in M_i and the sequences $\{C_i\}$ and $\{M_i - C_i\}$ converge to C_0 and $\mathrm{cl}(M_0 - C_0)$. Since the sequence $\{g_i(A_0)\}$ converges completely regularly to A_0 and the convergence of $\{C_i\}$ is h-2-regular at each point of $\operatorname{int} C_0$, the lemma and theorems of Section 3 up to and including Theorem 3.10 can now be applied to arcs, discs and annuli in C_0 whose boundaries lie in A_0. This proves Lemma 4.1.

LEMMA 4.2. *If A_0 is an annulus in M_0 with boundary curves J_0' and J_0'' such that $A_0 \cap K_0 = J_0' \cup J_0''$, then there is a sequence $\{A_i\}$ of annuli converging completely regularly to A_0 such that for each i, $K_i \cap A_i = \mathrm{bdry}\, A_i = g_i(J_0' \cup J_0'')$.*

Proof. If ϵ is a positive number, there are regular ϵ-neighborhoods A_0' and A_0'' of J_0' and J_0'' in K_0 which are annuli with boundary curves Z_0', X_0' and Z_0'', X_0'' respectively. Also, there are annuli B_0 and C_0 in M_0 such that $B_0 \cap K_0 = \mathrm{bdry}\, B_0 = X_0' \cup X_0''$, $C_0 \cap K_0 = \mathrm{bdry}\, C_0 = Z_0' \cup Z_0''$, and the torus $B_0 \cup C_0 \cup A_0' \cup A_0''$ bounds a 3-manifold with boundary, T_0, in M_0 which contains A_0 in its interior. It follows from a slight extension of Lemma 2.13 that there are sequences of annuli $\{B_i\}$ and $\{C_i\}$ converging strongly to B_0 and C_0 such that for each i, $B_i \cup C_i \subset M_i$, $B_i \cap K_i = \mathrm{bdry}\, B_i = g_i(X_0 \cup X_0'')$ and $C_i \cap K_i = \mathrm{bdry}\, C_i = g_i(Z_0' \cup Z_0'')$. As in the proof of Lemma 2.15, for sufficiently large i, the torus $C_i \cup B_i \cup g_i(A_0' \cup A_0'')$ bounds a compact 3-manifold with boundary, T_i in M_i and the sequences $\{T_i\}$ and $\{M_i - T_i\}$ converge to T_0 and $\mathrm{cl}(M_0 - T_0)$.

If t is an arc in T_i such that $t \cap \mathrm{bdry}\, T_i$ is the union of the endpoints of t and one of these lies in $g_i(A_0')$, the other in $g_i(A_0'')$, then t will be said to be unknotted in T_i provided that it lies in an annulus A_i^* in T_i such that $A_i^* \cap \mathrm{bdry}\, T_i = \mathrm{bdry}\, A_i^*$, $g_i(A_0') \cap A_i^*$ is deformable into $g_i(X_0')$ (isotopically) in $g_0(A_0')$ and $A_i^* \cap g_i(A_0'')$ is isotopically deformable into $g_i(X_0'')$ in $g_i(A_0'')$. With this definition and the fact that $\{g_i(A_0' \cup A_0'')\}$ converges completely regularly to $A_0' \cup A^{*\prime\prime}$, the proofs in Section 3 up to and including that of Theorem 3.10 can be adapted to yield a proof of Lemma 4.2.

LEMMA 4.3. *If x_0 is a 3-cell in M_0 such that $x_0 \cap K_0$ is a compact 2-manifold with boundary ($M_0 - \operatorname{int} x_0$ is thus a compact 3-manifold with boundary), then there is a sequence $x_1, x_2, \cdots$ of 3-cells converging completely regularly to x_0 such that for each i, x_i lies in M_i and $x_i \cap K_i = g_i(x_0 \cap K_0)$ and $\{M_i - \operatorname{int} x_i\}$ converges h-2-regularly to $M_0 - \operatorname{int} x_0$ and*

$$\{\mathrm{bdry}(M_i - \operatorname{int} x_i)\}$$

converges completely regularly to $\mathrm{bdry}(M_0 - \operatorname{int} x_0)$.

Proof. The lemma is proved by induction on the number of components of $x_0 \cap K_0$. Suppose, first, that $x_0 \cap K_0$ is connected. Then the closure, E, of each component of bdry $x_0 - (x_0 \cap K_0)$ is a disc bounded by a simple closed curve J. Denote these by $E_1, J_1, \cdots, E_k, J_k$. Then it follows from Lemma 4.1 that for each j there exists a sequence $\{E_{ij}\}_i$ of discs converging completely regularly to E_i such that for each i, $E_{ij} \subset M_i$ and $E_{ij} \cap K_i = \text{bdry}\, E_{ij} = g_i(J_j)$. For sufficiently large i, the 2-sphere, $g_i(x_0 \cap K_0) \cup \cup E_{ij}$ bounds a 3-cell x_i in M_i. The sequences $\{x_i\}$ and $\{M_i - \text{int}\, x_i\}$ converge to x_0 and $M_0 - \text{int}\, x_0$. That the convergence is h-2-regular, and hence completely regular, for $\{x_i\}$ now follows from the complete regularity of the convergence of $\{\text{bdry}\, x_i\}$ to bdry x_0 and an application of Lemma 3.11, which applies to this case as well as the case in which each M_i is a 3-cell.

Assume Lemma 4.3 to be true for all x_0, M_0 and K_0 for which $x_0 \cap K_0$ has fewer than r components and suppose that $x_0 \cap K_0$ here has r components. There are mutually exclusive simple closed curves J_0' and J_0'' in bdry $x_0 - (x_0 \cap K_0)$ bounding mutually exclusive discs D_0' and D_0'' whose interiors lie in int x_0 such that each of D_0' and D_0'' separates $x_0 \cap K_0$ in x_0 and $J_0' \cup J_0''$ bounds an annulus A_0 in bdry $x_0 - (x_0 \cap K_0)$. Denote by x_0', x_0'' and y_0 the 3-cells which are the closures of the sets into which $D_0' \cup D_0''$ separates x_0, D_0' and D_0'' belonging to x_0' and x_0'' respectively, $D_0' \cup D_0''$ belonging to y_0. Each of $x_0' \cap K_0$ and $x_0'' \cap K_0$ has fewer than r components so that there are sequences $\{x_i'\}$ and $\{x_i''\}$ of 3-cells converging completely regularly to x_0' and x_0'' such that for each i, $x_i' \cup x_i'' \subset M_i$ and $(x_i' \cup x_i'') \cap K_i = g_i[(x_0' \cup x_0'') \cap K_0]$ and the sequence $\{M_i - \text{int}(x_i' \cup x_i'')\}$ converges h-2-regularly to $M_0 - \text{int}(x_0' \cup x_0'')$, the convergence of these sets being completely regular on their boundaries. This can be done by first constructing the x_i' and then applying the lemma to $M_0 - \text{int}\, x_0'$ and x_0''.

Extend g_i to a δ_i-homeomorphism g_i' of $K_0 \cup x_0' \cup x_0''$ onto $K_i \cup x_i' \cup x_i''$, the sequence $\{\delta_i\}$ converging to 0. It follows from Lemma 4.2 applied to $M_0 - \text{int}(x_0' \cup x_0'')$ that there is a sequence of annuli, $\{A_i\}$ converging compeletely regularly to A_0 such that for each i, int $A_i \subset M_i - (x_i' \cup x_i'')$ and bdry $A_i = g_i'(J_0' \cup J_0'')$. For sufficiently large i,

$$(\text{bdry}\, x_i' \cup \text{bdry}\, x_i'' \cup A_i) - \text{int}\, g_i'(D_0' \cup D_0'')$$

bounds a 3-cell x_i in M_i and the sequence $\{x_i\}$ satisfies the conditions required of it by the lemma.

THEOREM 4.4. *The sequence $\{M_i\}$ converges to M_0 completely regularly.*

Proof. There is a cellular decomposition G of M_0 in the sense that each

element of G is a polyhedral 3-cell and the intersection of each element of G with the union of any number of elements of the collection consisting of K_0 and the elements of G is a compact 2-manifold with boundary, which may be empty. Such a decomposition is described by Bing in [2], p. 17. If x_0 is an element of G, then $M_0 - \operatorname{int} x_0$ is a compact 3-manifold with boundary, perhaps not connected, and the decomposition G^* of $M_0 - \operatorname{int} x_0$ consisting of the elements of $G - x_0$ is a cellular decomposition in the sense described above.

Arrange the elements of G in a sequence $x_1, x_2, \cdots, x_k$ with the property that for each j, x_j intersects the boundary of $M_0 - \bigcup x_r$, $r < j$. Since G is a cellular decomposition, this is possible. There is a sequence $\{x_{1j}\}$ of 3-cells converging completely regularly to x_1 such that for each i, $x_{1i} \cap K_i = g_i(x_1 \cap K_0)$ and $x_{1i} \subset M_i$. Extend g_i to an ϵ_{1i}-homeomorphism g_{1i} of $K_0 \cup x_1$ onto $K_i \cup x_{1i}$, the sequence $\{\epsilon_{1i}\}$ converging to 0. The sequence $\{M_i - \operatorname{int} x_{1i}\}$ converges h-2-regularly to $M_0 - \operatorname{int} x_0$ and since $x_2, \cdots, x_k$ is a cellular decomposition of $M_0 - \operatorname{int} x_0$, the above process may be applied to $M_i - \operatorname{int} x_{1i}$ to obtain a sequence $\{x_{2i}\}$ of 3-cells converging completely regular to x_2 such that for each i, $x_{2i} \subset M_i - \operatorname{int} x_{1i}$ and

$$x_{2i} \cap \operatorname{bdry}(M_i - \operatorname{int} x_{ij}) = g_{1i}(x_2 \cap \operatorname{bdry}(M_0 - \operatorname{int} x_1)).$$

The homeomorphism g_{1i} may be extended to an ϵ_{2i}-homeomorphism g_{2i} of $K_0 \cup x_1 \cup x_2$ onto $K_i \cup x_{1i} \cup x_{2i}$. The theorem is now proved by repeating this process for each $j \leqq k$.

A direct consequence of this theorem is the main theorem of this section.

THEOREM 4. 5. *If f is an h-2-regular mapping of a metric space X onto a metric space Y such that each inverse under f is a compact 3-manifold with boundary which is imbeddable in E_3 and the boundaries of the inverses under f are mutually homeomorphic, then f is completely regular.*

5. Some consequence of the preceding theory. In [6], Theorem 2, there was proved in a silghtly more general form the

THEOREM A. *Suppose that X is a complete metric space, Y is a metric space with finite covering dimension and f is a completely regular mapping of X onto Y such that* (1) *for each point p of Y there is a homeomorphism f_p of the $(i+1)$-cell R^{i+1}, with boundary S^i, onto $f^{-1}(p)$ and* (2) *there is a homeomorphism h of $\bigcup f_p(S^i)$, $p \in Y$, onto the direct product $Y \times S^i$ such that the diagram*

$$\begin{tikzcd} \cup f_p(S^i) \arrow[r, "h"] \arrow[dr, "f"'] & Y \times S^i \arrow[d, "\pi"] \\ & Y, \end{tikzcd}$$

where π *is the projection map, is commutative. Then there is a homeomorphism* h^* *of* X *onto the direct product* $Y \times R^{i+1}$ *which extends* h *and is such that the diagram*

$$\begin{tikzcd} X \arrow[r, "h^*"] \arrow[dr, "f"'] & Y \times R^{i+1} \arrow[d, "\pi"] \\ & Y \end{tikzcd}$$

is commutative.

This theorem and its proof yield the following typical result from [7].

THEOREM B. *If* f *is a* 0*-regular mapping of the complete metric space* X *onto the finite (covering) dimensional space* Y *such that each inverse* f *is homeomorphic to the compact 2-manifold with boundary,* M*, then* (X, f, y) *is a locally trivial fibre space. If* Y *is locally compact, separable and contractible, then* X *is homeomorphic to* $Y \times M$*, where* f *corresponds to the projection mappings of* $Y \times M$ *into* Y.

A consequence of Theorem A and Theorems 3.13 and 4.5 is

THEOREM 5.1. *If* f *is an* h*-2-regular mapping of the complete metric space* X *onto the finite (covering) dimensional space* Y *such that each inverse under* f *is a 3-cell,* R^3*, with boundary* S^2*, then* (X, f, Y) *is a locally trivial fibre space. If* Y *is locally compact, separable and contractible, then* X *is homeomorphic to* $Y \times R^3$*, where* f *corresponds to the projection map of* $Y \times R^3$ *onto* Y*. If there is a homeomorphism* h *of* $\cup \operatorname{bdry} f^{-1}(p)$, $p \in Y$, *onto* $Y \times S^2$ *such that the diagram*

$$\begin{tikzcd} \cup \operatorname{bdry} f^{-1}(p) \arrow[r, "h"] \arrow[dr, "f"'] & Y \times S^2 \arrow[d, "\pi"] \\ & Y \end{tikzcd}$$

is commutative, then h *may be extended to a homeomorphism* h^* *of* X *onto* Y *such that the diagram*

$$\begin{tikzcd} X \arrow[r, "h^*"] \arrow[dr, "f"'] & Y \times R^3 \arrow[d, "\pi"] \\ & Y \end{tikzcd}$$

is commutative.

Proof. The mapping f is completely regular. Thus $f \mid \cup \operatorname{bdry} f^{-1}(p)$ is completely regular and $(\cup \operatorname{bdry} f^{-1}(p), f, Y)$ is, by Theorem B, a locally trivial fibre space. The first part of the theorem now follows from Theorem A, as does the third part. If Y is locally compact, separable and contractible, then Theorem B implies that the hypothesis for the third part of the present theorem is fulfilled. Hence X is homeomorphic to $Y \times R^3$, f corresponding to the projection map.

The proof of Theorem A depends strongly on the fact that the space of homeomorphisms of a 3-cell onto itself leaving its boundary pointwise fixed is LC^n for each n[1]. (A space X is LC^n if for each point x in X and each $\epsilon > 0$ there is a $\delta > 0$ such that every mapping of a k-sphere, $k \leqq n$, into $S(x, \delta)$ is homotopic to 0 in $S(x, \epsilon)$.) It is clear from the proof of Theorem A that if M is a compact 3-manifold with boundary and (1) the space of homeomorphisms of M onto itself leaving bdry M pointwise fixed is locally connected (LC^0) and (2) for each positive number ϵ there is a positive number δ such that every δ-homeomorphism of bdry M onto itself can be extended to an ϵ-homeomorphism of M onto itself, then Theorem A remains true for one-dimensional Y if R^{i+1} is replaced by M and S^i is replaced by the boundary of M. (See the proof of Theorem 3 of [7].) Proofs of these two facts are included here. The proofs of Lemmas 5.4 and 5.3 were suggested by J. H. Roberts [13]. Lemma 5.4 has been proved by Sanderson. His proof is not yet published, but see [14]. For further results, see [8]. See also the recent work of Kister and Fisher to appear in the Transactions of the Ameriacn Mathematical Society ([3], [4], and [9]).

LEMMA 5.2. *If M is a compact 3-manifold with boundary, then for each positive number ϵ there is a positive number δ such that every δ-homeomorphism of* bdry M *onto itself can be extended to an ϵ-homeomorphism of M onto itself.*

Proof. Denote by $C_1, C_2, \cdots$ the components of the boundary of M. Each C_i is a compact 2-manifold. There is a homeomorphism h of bdry $M \times I$, where I is the unit interval, into M such that for each point p in bdry M,

$h(p, I)$ has diameter less than $\epsilon/3$ and $h(p, 1) = p$. (See [11].) It follows from Theorem 1 of [7] that the space of homeomorphisms of bdry M onto itself is locally connected. Denote by H this space of homeomorphisms and by i its identity. There is a positive number δ such that if $f \in S(i, \delta)$ in H, then there is a mapping F of I into $S(i, \epsilon/3)$ such that $F(0) = i$ and $F(1) = f$. If $y \in I$ and $q = h(p, y)$, let $f^*(q)$ denote $h(F(y)(p), y)$. If $q \in M - h(\text{bdry}\, M \times I)$, let $f^*(q) = q$. Since $d(q, p) < \epsilon/3$, $d(p, F(y)(p)) < \epsilon/3$ and $d[F(y)(p), h(F(y)(p), y)] < \epsilon/3$, and $f^*(q) = h(F(1)(q), 1) = h(f(q), 1) = f(q)$. Thus f^* extends f and the lemma is proved.

LEMMA 5.3. *Suppose that K is a 3-manifold with boundary, K_2 is a polyhedral 3-cell in K which intersects* bdry K *in a 2-manifold or not at all and K_2' is a polyhedral 3-cell in K_2 such that* $K_2' \cap \text{bdry}\, K_2 \subset \text{int}(K_2 \cap \text{bdry}\, K)$. *If ϵ is a positive number, then there is a positive number δ such that if f is a piecewise linear δ-homemorphism of K onto itself leaving* bdry K *pointwise fixed, then there is a piecewise linear ϵ-homeomorphism g^* of K onto itself which is the identity outside K_2, is g on K_2' and leaves* bdry K *pointwise fixed.*

Proof. Denote by K^* the compact 3-manifold obtained by "sewing" a homeomorphic image of K to K along its boundary. Let K_1 denote $\text{cl}(K - K_2)$ and U a polyhedral neighborhood of $K_1 \cap K_2$ in K such that $U \cap K_2' = 0$, U is a 3-manifold with boundary, $\text{cl}(\text{int}\, K \cap \text{bdry}\, U)$ is a 2-manifold with boundary and $U \cap K_2$ is a 3-manifold with boundary. Let V denote a polyhedral neighborhood of $\text{bdry}(U \cap K_2)$ in K^* such that $V \cap K_2' = 0$. If ϵ is a positive number, it follows from Moise's lemma on the fitting together of homeomorphisms [11] that there is a positive number δ such that if t_1 is a piecewise linear δ-homomorphism of $(U \cap K_2) \cup V$ into K^* and t_2 is a piecewise linear δ-homeomorphism of $K^* - (U \cap K_2)$ into K^*, then there is a piecewise linear ϵ-homeomorphism t of K^* onto itself such that $t \mid (K_2 \cap U - V = t_1$ and $t \mid K^* - (U \cap K_2) = t_2$.

Suppose that g is a piecewise linear δ-homeomorphism of K onto itself (hence into K^*) leaving bdry K pointwise fixed. Denote by t_1 the identity homeomorphism of $(U \cap K_2) \cup V$ into K^* and by t_2 the δ-homeomorphism of $K^* - (U \cap K_2)$ into K^* which is identical to g when restricted to $K_2 - (U \cap K_2)$ and is the identity when restricted to $K_1 \cup (K^* - K)$. Then there is a piecewise linear ϵ-homeomorphism f of K^* onto itself such that $f \mid (U \cap K_2) - V$ is the identity and $f \mid K^* - (U \cap K_2) = t_2$. Thus $g^* = f \mid K$ is such that $g^* \mid K_2' = g \mid K_2'$ and $g^* \mid K_1$ is the identity, since bdry $K \subset \text{cl}(K^* - (K_2 \cap U)$, and $g^* \mid \text{bdry}\, K$ is the identity on $K_1 \cup (K^* - K)$.

LEMMA 5.4. *If K is a compact 3-manifold with boundary, then the space of homeomorphisms of K onto itself leaving* bdry K *pointwise fixed is locally connected.*

Proof. Let G be a cellular decomposition of K as described in Section 4. Induction will be used on the number of elements of G. If G has just one element, then K is a 3-cell and the lemma follows from a well known theorem of Alexander [1]. Suppose the lemma to be true for all compact 3-manifolds with boundary which have a cellular decomposition with fewer than k elements and that G, here, has k elements. Denote by X an element of G and by X' a 3-cell in X such that $X' \cap \operatorname{bdry} X \subset \operatorname{int}(X \cap \operatorname{bdry} K)$ and $\operatorname{cl}(K - X')$ is homeomorphic to $\operatorname{cl}(K - X)$. Denote by $H(K)$, $H(X)$, and $H(\operatorname{cl}(K - X'))$ the spaces of homeomorphisms of K, X, and $\operatorname{cl}(K - X')$ onto themselves leaving the boundaries pointwise fixed. (If f and g are homeomorphisms in one of these spaces, $d(f, g) = \operatorname{lub}[d(f(x), g(x))]$.) Suppose ϵ is a positive number. There is a positive number δ' such that (1) if f is a $2\delta'$-homeomorphism in $H(X)$, there is a mapping F of I into $H(X)$ such that $F(0) = i \mid X$, $F(1) = f$ and $F(t)$ move no point as much as $\epsilon/2$ and (2) if f is a $2\delta'$-mapping in $H(\operatorname{cl}(K - X'))$, there is a mapping F of I into $H(\operatorname{cl}(K - X'))$ such that $F(0) = i \mid \operatorname{cl}(K - X')$, $F(1) = f$ and $F(t)$ moves no point as much as $\epsilon/2$. Statement (2) follows from the induction hypothesis and the fact that the elements of $G - (X)$ form a cellular decomposition of $\operatorname{cl}(K - X)$, which is homeomorphic to $\operatorname{cl}(K - X')$, with fewer than k elements. Also, there exists a positive number δ such that if g is a piecewise linear δ-homeomorphism in $H(K)$, then there is a piecewise linear δ'-homeomirphism g^* such that $g^* \mid X' = g \mid X'$ and $g^* \mid K - X = i \mid K - X$.

Suppose then that g is a piecewise linear δ-homeomorphism in $H(K)$. There is a δ'-homeomorphism g^* such that $g^* \mid X' = g \mid X'$ and $g^* \mid K - X = i \mid K - X$. Clearly $g^* \mid X$ is an element of $H(X)$. Thus there is a mapping F of I into $H(K)$ such that $F(0) = i$, $F(1) = g^*$ and $F(t)$ is an $\epsilon/2$-homeomorphism which moves no point of $K - X$. The mapping $g^{*-1}g \mid X' = i \mid X'$ and therefore $g^{*-1}g \mid \operatorname{cl}(K - X')$ is an element of $H(\operatorname{cl}(K - X'))$ and moves no point as much as $2\delta'$. Hence there is a mapping F^* of I into $H(K)$ such that $F^*(0) = i$, $F^*(1) = g^{*-1}g$ and $F^*(t)$ is an $\epsilon/2$-homeomorphism which moves no point of X'. Let $Z(t)$ denote $F(t)F^*(t)$. Then $Z(0) = i$ and $Z(1) = g^*g^{*-1}g = g$. Furthermore, each $Z(t)$ is an ϵ-homeomorphism. Thus each piecewise linear δ-homeomorphism is connected to the identity by an arc of diameter less than 2ϵ.

Now let g be any $\delta/2$-homeomorphism in $H(K)$ and let $\{\delta_i\}$ and $\{\epsilon_i\}$ be

decreasing sequences of positive numbers converging to 0 such that each piecewise linear $2\delta_i$-homeomorphism in $H(K)$ may be joined to the identity by an arc of ϵ_i-homeomorphisms. Let K^* be as described in the proof of Lemma 5.3 and let U denote a polyhedral neighborhood of bdry K in K^*. Denote by g^* the homeomorphism of K^* onto itself such that $g^* \mid K = g$ and $g^* \mid K^* - K = i$. From the lemma on the fitting together of homeomorphisms [11] it follows that there is a positive number δ_i' such that if f_i and f_i' are piecewise linear δ_i'-approximations to $g^* \mid K \cup U$ and $g^* \mid \mathrm{cl}(K^* - K)$, then there is a piecewise linear δ_i-approximation, g_i^* to g^* such that $g^* \mid K^* - K = f_i'$ and $g_i^* \mid K - U = f_i$. From the theorem of Moise on the approximation of homeomorphisms by piecewise linear ones [11] it follows that there is a piecewise linear δ_i-approximation (homeomorphism) f_i to $g^* \mid K \cup U$. Denote by f_i' the mapping $i \mid \mathrm{cl}(K^* - K)$. Then there is a piecewise linear δ_i-approximation g_i^* to g^* such that $g_i^* \mid K^* - K = i$. Then $g_i = g_i^* \mid K$ is an element of $H(K)$ and is a piecewise linear δ_i-approximation to g.

It may be assumed that $d(g_i, g_{i+1}) < 2\delta_i$. Then $d(i, g_{i+1}g_i^{-1}) < 2\delta_i$, so that there is a mapping F_i of I into $H(K)$ such that $F_i(0) = i$ and $F_i(1) = g_{i+1}g_i^{-1}$ and $d(i, F_i(t)) < \epsilon_i$. Then the mapping F_i^* of I into $H(K)$ defined by the equation $F_i^*(t) = F_i(t)g_i$ is such that $F_{i_0}(0) = g_i$, $F_i^*(1) = g_{i+1}$ and $d(g_i, F_i^*(t)) < \epsilon_i$. Since the sequence $\{g_i\}$ converges to g, $\{\epsilon_i\}$ converges to 0 and there is an arc of diameter less than $2\epsilon_i$ connecting g_i to g_{i+1}, it has been shown that there is an arc of diameter less than $\delta/2$ connecting g to a piecewise linear homeomorphism g' in $H(K)$. Since g' can be connected to i by an arc of diameter less than 2ϵ, the local connectedness of $H(K)$ at the identity is established. Since $H(K)$ can be given a group structure, it is locally connected at all points and the lemma is proved.

THEOREM 5.5. *If f is an h-2-regular mapping of a complete metric space X onto a one-dimensional space Y such that each inverse under f is homeomorphic to the compact 3-manifold M with boundary which is imbeddable in E^3, then (X, f, Y) is a locally trivial fibre space. If Y is locally compact, separable and contractible, then X is homeomorphic to $M \times Y$, where f corresponds to the projection map of $M \times Y$ onto Y. If there is a homeomorphism f of $\cup$ bdry $f^{-1}(p)$, $p \in Y$, onto bdry $M \times Y$ such that the diagram*

$$\begin{tikzcd}
\cup\, \mathrm{bdry}\, f^{-1}(p) \arrow[r, "h"] \arrow[dr, "f"'] & \mathrm{bdry}\, M \times Y \arrow[d, "\pi"] \\
 & Y
\end{tikzcd}$$

is commutative, then, if the space of homeomorphisms of M onto itself leaving its boundary pointwise fixed is connected, h may be extended to a homeomorphism h^* of X onto $M \times Y$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "h^*"] \arrow[dr, "f"'] & M \times Y \arrow[d, "\pi"] \\ & Y \end{tikzcd}$$

is commutative.

Proof. It follows from Theorem 4.5 that f is completely regular. Thus, by Theorem B, $(\cup \operatorname{bdry} f^{-1}(p), f, Y)$ is a locally trivial fibre space. The theorem now follows from Lemmas 5.2 and 5.4 and a slight modification of the proof of Theorem A (see Thesorem 3 of [7]).

Note. In view of Theorem 4.5, if Y is connected, it is only necessary to assume here that each $f^{-1}(p)$ is homeomorphic to a 3-manifold with boundary which is imbeddable in E^3 and that the boundaries of the inverses are mutually homeomorphic. Also, the restriction on Y that it be one-dimensional is now known to be unnecessary. If the space of homeomorphisms of M onto itself leaving bdry M pointwise fixed is LC^n, then Y may be taken to be $(n+1)$-dimensional, as the proof of Theorem A indicates. That this is true for each n will be proved in a later paper [8].

6. Some weakening of the hypotheses in earlier theorems. In this section, the notation of section 2, unless specific modification are stated, will be used.

THEOREM 6.1. *If the sequence $\{M_i\}$ converges to M_0 h-1-regularly, then it converges h-2-regularly.*

Proof. Note first that none of the proofs in Section 3 require more than h-1-regularity. Suppose that P is a point in $\operatorname{int} M_0$ and that ϵ is a positive number. Let S_0 denote a 2-sphere in $\operatorname{int} M_0$ bounding a 3-cell in $\operatorname{int} M_0$ whose interior contains P and has diameter less than ϵ. Then it follows from Lemma 2.15 that there is a sequence $\{S_i\}$ of 2-spheres converging strongly to S_0 such that for each i, S_i bounds a 3-cell A_i in M_i and the sequences $\{A_i\}$ and $\{M_i - \operatorname{int} A_i\}$ converges to A_0 and $M_0 - \operatorname{int} A_0$. For sufficiently large i, A_i is a subset of $S(P, \epsilon)$. Let δ be a positive number such that for each i,

$S(P,\delta) \cap M_i \subset \operatorname{int} A_i$. Then every mapping of a 2-sphere into $S(P,\delta) \cap M_i$ is homotopic to 0 in A_i and consequently in $S(P,\epsilon) \cap M_i$.

If P is a point of K_0, let C_0 denote a disc in K_0 whose interior contains P and has diameter less than ϵ. Let D_0 denote a disc in M_0 such that $D_0 \cap K_0 = \operatorname{bdry} D_0 = \operatorname{bdry} C_0$ and the 2-sphere $D_0 \cup C_0$ bounds a 3-cell in M_0 of diameter less than ϵ. From Lemma 2.13 and Theorem 2.14 it follows that there is a sequence $\{D_i\}$ of discs converging to D_0 such that for each i, $\{D_i\}$ lies in M_i, $D_i \cap K_i = \operatorname{bdry} D_i$ which is the boundary of a disc C_i in K_i, $C_i \cup D_i$ bounds a 3-cell A_i in M_i and the sequences $\{A_i\}$ and $\{M_i - \operatorname{int} A_i\}$ converge to A_0 and $M_0 - \operatorname{int} A_0$. A repetition of the argument in the foregoing paragraph now demonstrates that the convergence is h-2-regular at each point of K_0 and Theorem 6.1 is proved.

THEOREM 6.2. *If the sequence $\{M_i\}$ of compact 3-manifolds with boundary converges h-2-regularly to the compact 3-manifold with boundary M_0 and each M_i is imbeddable in E^3, then for sufficiently large i,* $\operatorname{bdry} M_i$ *is homeomorphic to* $\operatorname{bdry} M_0$.

Proof. As before, denote $\operatorname{bdry} M_i$ by K_i. It follows from Lemma 2.9 that if C_0 is a component of K_0, then there is a sequence $\{C_i\}$ of components of $\{K_i\}$ converging strongly to C_0. It will first be proved that this convergence is completely regular. Suppose that J_0 is a simple closed curve bounding a disc A_0 in C_0. The argument for Theorem 2.14 may be applied to prove that there is a sequence $\{J_i\}$ of simple closed curves converging strongly to J_0 such that for each i, J_i bounds a compact 2-manifold with boundary, A_i, in C_i, the sequences $\{A_i\}$ and $\{C_i - \operatorname{int} A_i\}$ converging to A_0 and $C_0 - \operatorname{int} A_0$. Suppose that A_i is a disc with handles. Then there is a pair of simple closed curves, x_i and y_i, in A_i which cross each other and have only one point in common. It follows from the 1-regularity of the convergence of M_i to M_0 that if A_0 has sufficiently small diameter, x_i and y_i bound singular discs B_0 and D_i in M_i. Dehn's Lemma implies that B_i and D_i may be taken to be non-singular and such that $B_i \cap K_i = x_i$ and $D_i \cap K_i = y_i$. Small adjustments may be made in D_i and B_i so that each component of $B_i \cap D_i$ other than $x_i \cap y_i$ is a simple closed curve. These components may be removed in a manner described earlier in this paper. This process leaves two discs B_i' and D_i' such that $B_i' \cap D_i' = x_i \cap y_i$, $B_i' \cap K_i = x_i$ and $D_i' \cap K_i = y_i$. This is clearly impossible. Hence for sufficiently large i, A_i is a disc. It follows from the proof of Theorem 2.14 that the convergence of $\{C_i\}$ to C_0 is h-1-regular and hence completely regular.

Theorem 6.2 will now be proved when it is shown that no point of C_0

is a limit point of $\cup(K_i - C_i)$ and that no point of $\operatorname{int} M_i$ is a limit point of $\cup K_i$. Let ϵ be a positive number, P a point of C_0 and A_0 a disc in C_0 of diameter less than ϵ whose interior contains P. There is a disc D_0 in M_0 such that $D_0 \cap K_0 = \operatorname{bdry} D_0 = \operatorname{bdry} A_0$ and the 2-sphere $D_0 \cup A_0$ bounds a 3-cell N_0 in M_0 of diameter less than ϵ. It follows from the proofs of Lemma 2.13 and Theorem 2.14 that there are sequences $\{A_i\}$ and $\{D_i\}$ converging to A_0 and D_0 such that for each i, A_i is a disc in C_i, D_i is a disc in M_i, $D_i \cap K_i = \operatorname{bdry} D_i = \operatorname{bdry} A_i$ and that $M_i - (D_i \cup A_i)$ has two components, the closure of one denoted by N_i, such that the sequences $\{N_i\}$ and $\{M_i - \operatorname{int} N_i\}$ converge to subsets of N_0 and $M_0 - \operatorname{int} N_0$. No component of $K_i - C_i$ intersects D_i so that if P is a limit point of $\cup(K_i - C_i)$, then there is a sequence $\{R_{n_i}\}_i$ of components of $\{K_{n_i}\}$ converging to a subset of N_0. However, it follows from the h-2-regularity of the convergence that for sufficiently large i and small ϵ, $D_i \cup A_i$ bounds a singular 3-cell, N_i', in M_i which, since M_i is imbeddable in E^3, contains a non-singular 3-cell, N_i'', in M_i whose boundary is $D_i \cup A_i$. But $R_{n_i} \subset N_{n_i}''$ for large i, which contradicts the fact that $R_i \subset K_i$.

If P is a point of $\operatorname{int} M_0$, let S_0 be a 3-sphere in $\operatorname{int} M_0$ bounding a 3-cell A_0 in $\operatorname{int} M_0$ whose interior contains P and has diameter less than ϵ. It follows from Lemma 2.15 that there is a sequence $\{S_i\}$ of 2-spheres converging strongly to S_0 such that for each i, $S_i \subset \operatorname{int} M_i$ and $M_i - S_i$ has two components, the closure of one denoted by A_i, such that the sequences $\{A_i\}$ and $\{M_i - \operatorname{int} A_i\}$ converge to subsets of A_0 and $M_0 - \operatorname{int} A_0$. The argument in the foregoing paragraph may now be used to prove that P is not a limit point of $\cup K_i$. This completes the proof of Theorem 6.2.

It is easy to find examples to show that Theorem 6.1 is not true if regularity is assumed only in dimensions 0 and 2 and that Theorem 6.2 is not true if only h-1-regularity is assumed. The theorems in this section demonstrate that the mapping f in Theorems 5.1 and 5.5 need only be h-1-regular and that under the hypothesis of h-1-regularity and the connectedness of Y, the first two parts of Theorem 5.5 remain true if each $f^{-1}(p)$ is only assumed to be a compact 3-manifold imbeddable in E^3.

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BASIC REPRESENTATIONS OF COMPLETELY SIMPLE SEMIGROUPS.*[1]

By A. H. Clifford.

1. Introduction. In a previous paper [1], the author discussed the theory of representations of a completely simple semigroup S by matrices over a field Ω. According to the fundamental theorem of Rees [2], S is isomorphic with, and hence may be taken to be, a regular matrix semigroup over a group with zero. It was shown in [1] that every representation $\mathfrak{T}^*$ of S induces a representation $\mathfrak{T}$ of G; we call $\mathfrak{T}^*$ an extension of $\mathfrak{T}$ to S.

A given representation $\mathfrak{T}$ of G may not be extendible to a representation $\mathfrak{T}^*$ of S; but if it is so extendible, then the extension $\mathfrak{T}_0^*$ of $\mathfrak{T}$ of least possible degree over Ω is uniquely determined by $\mathfrak{T}$ to within equivalence. We call $\mathfrak{T}_0^*$ the basic extension of $\mathfrak{T}$, and by a basic representation of S we shall mean one that is the basic extension to S of a representation of G. Any extension $\mathfrak{T}^*$ of a representation $\mathfrak{T}$ of G reduces (but does not in general decompose) into the basic extension $\mathfrak{T}_0^*$ of $\mathfrak{T}$ and null representations.

It is immediate from Theorems 4.1 and 6.1 of [1] that the mapping $\mathfrak{T} \rightarrow \mathfrak{T}_0^*$ is one-to-one (in the sense of equivalence) from the extendible representations of G to the basic representations of S. However, several questions concerning this correspondence were left unanswered. It was shown (Theorem 7.1) that if $\mathfrak{T}$ is irreducible, so is $\mathfrak{T}_0^*$, but the converse was left open. One of the main purposes of this note is to prove that the converse is true, and hence that *all the irreducible representations of S over Ω are obtained as the basic extensions to S of the extendible irreducible representations of G.*

In § 2 we show that the correspondence $\mathfrak{T} \rightarrow \mathfrak{T}_0^*$ preserves decomposition. In § 3 we show that it preserves reduction in a limited sense: the *non-null* irreducible constituents of $\mathfrak{T}_0^*$ are the basic extensions of the irreducible constituents of $\mathfrak{T}$. An example in § 4 shows that an extraneous null constituent can occur in $\mathfrak{T}_0^*$. (Thanks to W. D. Munn for pointing this out.)

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The author would like to take this opportunity of mentioning that the underlying ideas and methods of [1] should be attributed to Suschkewitsch [3]. He also proved the first part of Theorem 3.1. The remark in the introduction of [1] that Suschkewitsch "made considerable progress" was neither precise nor adequate.

2. Decomposition.

THEOREM 1. *A representation $\mathfrak{T}$ of G is extendible to S if and only if each of its indecomposable constituents is extendible. If $\mathfrak{T}$ is extendible, then the indecomposable constituents of the basic extension $\mathfrak{T}_0^*$ of $\mathfrak{T}$ are the basic extensions of the indecomposable constituents of $\mathfrak{T}$. In particular, $\mathfrak{T}_0^*$ is indecomposable if and only if $\mathfrak{T}$ is indecomposable.*

Proof. First let $\mathfrak{T}$ be an indecomposable representation of G which is extendible to S, and let $\mathfrak{T}_0^*$ be its basic extension to S. Suppose that $\mathfrak{T}_0^*$ could be decomposed into two representations $\mathfrak{R}$ and $\mathfrak{R}'$ of S each of lower degree than that of $\mathfrak{T}_0^*$. The restrictions of $\mathfrak{R}$ and $\mathfrak{R}'$ to G cannot share the indecomposable representation $\mathfrak{T}$ of G. Hence either $\mathfrak{R}$ or $\mathfrak{R}'$ is an extension to S of $\mathfrak{T}$, contrary to the fact that $\mathfrak{T}_0^*$ is the extension of $\mathfrak{T}$ to S of lowest possible degree.

Now suppose that $\mathfrak{T}$ is the direct sum $\mathfrak{T}' \oplus \mathfrak{T}''$ of two representations of G each of lower degree than that of $\mathfrak{T}$. According to Theorem 7.2 of [1], $\mathfrak{T}$ is extendible to S if and only if $\mathfrak{T}'$ and $\mathfrak{T}''$ are both extendible; and, if this is the case, then the basic extension of $\mathfrak{T}$ is equivalent to the direct sum of the basic extensions of $\mathfrak{T}'$ and $\mathfrak{T}''$. The rest of Theorem 1 then follows by an evident induction on the number of indecomposable constituents of $\mathfrak{T}$.

3. Reduction.

THEOREM 2. *Let $\mathfrak{T}$ be an extendible representation of G, and let $\mathfrak{T}^*$ be any extension of $\mathfrak{T}$ to S. Then the non-null irreducible constituents of $\mathfrak{T}^*$ are the basic extensions of the irreducible constituents of $\mathfrak{T}$. The basic extension $\mathfrak{T}_0^*$ of $\mathfrak{T}$ is irreducible if and only if $\mathfrak{T}$ is irreducible.*

Proof. Let $\mathfrak{T}$ be an extendible representation of G, and let $\mathfrak{T}^*$ be any extension of $\mathfrak{T}$ to S. Assume that $\mathfrak{T}$ reduces into representations $\mathfrak{T}'$ and $\mathfrak{T}''$ of G, each of lower degree over Ω than that of $\mathfrak{T}$. We proceed to show that $\mathfrak{T}^*$ reduces into two representations of S, one of which is an extension of $\mathfrak{T}'$ and the other of $\mathfrak{T}''$.

Equation (3.1) of [1] shows that the restriction of $\mathfrak{T}^*$ to G decomposes into the proper representation $\mathfrak{T}$ of G and a null representation of G. Let the corersponding decomposition of the representation space V of $\mathfrak{T}$ be $V = V_1 \oplus V_2$, where V_1 carries $\mathfrak{T}$ and V_2 carries the null representation. Let n be the dimension of V_1 over Ω, and t that of V_2. By Theorem 3.1 of [1], we may assume that the representing matrices of $\mathfrak{T}^*$ have the form

$$T^*[(a)_{i\kappa}] = \begin{pmatrix} T(p_{1i}ap_{\kappa 1}) & T(p_{1i}a)Q_\kappa \\ R_iT(ap_{\kappa 1}) & R_iT(a)Q_\kappa \end{pmatrix}, \tag{3.5}$$

where the R_i are $t \times n$ matrices and the Q_κ are $n \times t$ matrices satisfying

$$Q_\kappa R_i = T(p_{\kappa i}) - T(p_{\kappa 1}p_{1i}). \tag{3.7}$$

Let W_1 be the invariant subspace of V_1 which carries $\mathfrak{T}'$, so that $\mathfrak{T}''$ is carried by the factor-space V_1/W_1. Let W be the subspace of V consisting of all vectors w of V having the form

$$w = x + \sum_{i \in J} R_i x_i$$

with x and the x_i in W_1, and where the sum is finite, that is, all but a finite number of the x_i are the zero vector of W_1. Now R_i (for each i in J) may be regarded as a linear transformation of V_1 into V_2. Thus $w = x + y$ with x in W_1 and $y = \sum_i R_i x_i$ in V_2. It will be convenient in what follows to write w in block form $\begin{pmatrix} x \\ y \end{pmatrix}$ corresponding to (3.5). Since $W \subseteq W_1 \oplus V_2$, it is a proper subspace of V, and we proceed to show that it is invariant under $\mathfrak{T}^*$.

By direct calculation from (3.5), we have

$$T^*[(a)_{i\kappa}] \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

where, using (3.7) and $y = \sum_j R_j x_j$,

$$\begin{aligned} x' &= T(p_{1i}ap_{\kappa 1})x + T(p_{1i}a)\sum_j Q_\kappa R_j x_j \\ &= T(p_{1i}ap_{\kappa 1})x + \sum_j [T(p_{1i}ap_{\kappa j}) - T(p_{1i}ap_{\kappa 1}p_{1j})]x_j \end{aligned}$$

and

$$\begin{aligned} y' &= R_iT(ap_{\kappa 1})x + R_iT(a)\sum_j Q_\kappa R_j x_j \\ &= R_iT(ap_{\kappa 1})x + R_i\sum_j [T(ap_{\kappa j}) - T(ap_{\kappa 1}p_{1j})]x_j. \end{aligned}$$

Since x and all the x_i belong to W_1, and W_1 is invariant under $T(b)$ for

every b in G, it is clear that $x' \in W_1$. Since y' is seen to be of the form $R_i x''$ wtih x'' in W_1, it follows that $x' + y' \in W$, and so W is invariant under $\mathfrak{T}^*$.

Let $\mathfrak{R}'$ be the representation of S carried by the invariant subspace W of V constructed above, and let $\mathfrak{R}''$ be that carried by the factor space V/W, so that $\mathfrak{T}^*$ reduces into $\mathfrak{R}'$ and $\mathfrak{R}''$. Now $V = V_1 \oplus V_2$ and $W = W_1 \oplus W_2$, where $W_2 = W \cap V_2$ $(= \sum_{i \in J} R_i W_1)$. Hence

$$V/W = V_1/W_1 \oplus V_2/W_2.$$

The representation of G induced by $\mathfrak{R}'$ is the proper part of the representation of G carried by W. Since W_1 carries the proper representation $\mathfrak{T}'$ of G, while W_2 carries a null representation of G (since $W_2 \subseteq V_2$), it follows that $\mathfrak{R}'$ induces $\mathfrak{T}'$. Similarly, V_1/W_1 carries $\mathfrak{T}''$ and V_2/W_2 carries a null representation of G, and so $\mathfrak{R}''$ induces $\mathfrak{T}''$ in G. Hence $\mathfrak{R}'$ is an extension of $\mathfrak{T}'$, and $\mathfrak{R}''$ an extension of $\mathfrak{T}''$ to S.

By an evident induction on the number r of irreducible constituents $\mathfrak{T}_i$ of $\mathfrak{T}$, it is clear that $\mathfrak{T}^*$ reduces into r representations $\mathfrak{R}_i$ such that $\mathfrak{R}_i$ is an extension of $\mathfrak{T}_i$ $(i = 1, \cdots, r)$. By Theorem 6.2 of [1] $\mathfrak{R}_i$ reduces into the basic extension $\mathfrak{T}_{i0}^*$ of $\mathfrak{T}_i$ and (possibly) null representations. By Theorem 7.1, each $\mathfrak{T}_{i0}^*$ is irreducible. Hence the non-null irreducible constituents of $\mathfrak{T}^*$ are precisely $\mathfrak{T}_{10}^*, \cdots, \mathfrak{T}_{r0}^*$.

The final assertion of the theorem is immediate from the foregoing and Theorem 7.1.

4. Examples. In Theorem 2, let $\mathfrak{T}^*$ be the basic extension $\mathfrak{T}_0^*$ of $\mathfrak{T}$. One might expect that each irreducible constituent of $\mathfrak{T}_0^*$ is the basic extension of one of the irreducible constituents of $\mathfrak{T}$. The following example shows that $\mathfrak{T}_0^*$ may have an extraneous null constituent.

Let G be the cyclic group $\{e, a\}$ of order 2. Let S be the Rees 2×2 matrix semigroup over G with "sandwich" matrix $P = \begin{pmatrix} e & e \\ e & a \end{pmatrix}$. Let Ω be the integers mod 2. Let

$$T(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad T(a) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

It is found that the basic extension $\mathfrak{T}_0^*$ of $\mathfrak{T}$ to S has degree 3 over Ω, and reduces into two unit representations and one null representation of S.

According to Theorem 1, a representation $\mathfrak{T}$ of G is extendible to S if its indecomposable constituents are extendible. The following example shows that the irreducible constituents of $\mathfrak{T}$ may be extendible, yet $\mathfrak{T}$ is not extendible.

Let G, Ω, and $\mathfrak{T}$ be as in the previous example. Let N be the set of natural numbers, and let S be the Rees $N \times N$ matrix semigroup over G with sandwich matrix $P = (p_{ij})$ given by

$$p_{ij} = \begin{cases} e & \text{if } i = 1 \text{ or } j = 1, \\ a & \text{otherwise.} \end{cases}$$

Since the irreducible constituents of $\mathfrak{T}$ are just unit representations of G, they are trivially extendible to S. But $\mathfrak{T}$ itself is found not to be extendible.

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SUR LA THÉORIE DE LA VARIÉTÉ DE PICARD.*

par C. CHEVALLEY.

En hommage amical et respecteux au professeur Zariski.

Introduction. Soit U une variété complète. On appelle habituellement diviseurs de U les combinaisons linéaires formelles d'hypersurfaces de U à coefficients entiers, c'est-à-dire les cycles de codimension 1 sur U. Cartier a introduit récemment une autre notion de diviseur (cf. [1]) ; si U est normale, ce que nous supposerons ici, un diviseur au sens de Cartier est un cycle de codimension 1 qui est localement principal, i. e. qui coincide au voisinage de chaque point avec le diviseur d'une fonction. Si U est non singulière, les deux notions de diviseur sont équivalentes, mais il n'en est plus de même en général.

Nous nous proposons ici d'étendre aux diviseurs de Cartier (que nous appellerons désormais simplement diviseurs) la notion de variété de Picard, tout au moins dans le cas des variétés complètes U qui sont normales. W. L. Chow a observé que, pour la notion classique de diviseurs, la variété de Picard d'une variété quelconque U était identique à la variété de Picard de la variété d'Albanese de U. Nous montrons que ce résultat reste vrai pour les diviseurs au sens de Cartier à condition de remplacer la variété d'Albanese de U par ce que nous appelons sa variété d'Albanese stricte: alors que la variété d'Albanese résoud le problème relatif aux fonctions (non partout définies) sur U à valeurs dans des variétés abéliennes, la variété d'Albanese stricte résoud le problème correspondant relatif aux morphismes (partout définis) de U dans des variétés abéliennes. Comme une variété abélienne est non singulière, il n'y a pas de différence entre diviseurs au sens classique et diviseurs de Cartier sur une telle variété; nous aurions donc pu tenir pour acquies la notion de variété de Picard d'un variété abélienne pour établir le résultat cité ci-dessus. Nous avons préféré reprendre la question dans son ensemble, car, d'une part, la démonstration que nous donnons de l'existence d'une variété de Picard pour une variété abélienne est, croyons-nous, plus simple que les démonstrations déjà connues, et d'autre part, elle se poursuit dans un esprit tout différent.

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Nous utilisons systématiquement la notion de famille algébrique de diviseurs (ou de classes de diviseurs) d'une variété U paramétrée par une variété T: une famille algébrique de classes de diviseurs de U paramétrée par T est une application f de T dans le groupe des classes de diviseurs de U qui satisfait à certaines conditions. Supposons de plus que l'on ait $f(t_0)=0$ pour un certain point t_0 de T. La famille f définit un diviseur de U (ou plutôt d'une variété déduite de U par extension du corps de base) rationnel sur le corps $F(T)$ de la variété T, donc un point de la variété de Picard P de U rationnel sur $F(T)$, c'est-à-dire une fonction sur T à valeurs dans P; nous montrons que (au moins si T est normale) cette fonction est partout définie.

Chapitre I. Familles de diviseurs.

I. Diviseurs. Nous utiliserons la définition de la notion de diviseur sur une variété dûe à Cartier ([1]). Rappelons qu'il y a une correspondence biunivoque entre les diviseurs sur une variété U et les faisceaux cohérents d'idéaux fractionnaires principaux sur U (quand nous parlerons d'idéaux fractionnaires, il sera toujours sous-entendu qu'il s'agit d'idéaux $\neq \{0\}$). Si d est un diviseur, nous désignerons par A^d le faisceau qui correspond à d; pour tout faisceau F de groupes sur U, nous désignerons par F_x le groupe ponctuel de F en un point x de U; $A^d{}_x$ est donc un idéal fractionnaire principal pour l'anneau local $\mathfrak{o}(x)$ de x; tout générateur de cet idéal est appelé une *fonction de définition de d en x.* Toute fonction sur U qui est fonction de definition de d en x l'est aussi en tous les points d'un voisinage de x. Les diviseurs pour lesquels il existe une fonction qui est une fonction de définition en tous les points de U sont les *diviseurs principaux.* Si les fonctions de définion de d en x sont définies en x, on dit que d est *positif en x*; si d est positif en tous les points de U, on dit qu'il est *positif,* et on écrit $d \geqq 0$. Les section du faisceau A^d sont les fonctions numériques v telles que $v \in A^d{}_x$ pour tout $x \in U$; ces fonctions sont aussi dites être des *multiples de d.*

Les opérations du calcul sur les idéaux fractionnaires définissent des opérations sur les faisceaux d'idéaux fractionnaires. Ainsi, si A et A' sont des faisceaux d'idéaux fractionnaires, les symboles AA', $A+A'$ représentent des faisceaux d'idéaux fractionnaires dont les idéaux ponctuels en un point x sont $A_xA'_x$ et $A_x+A'_x$ respectivement; si d et d' sont des diviseurs, on a $A^dA^{d'}=A^{d+d'}$. Soit O le faisceaux des anneaux locaux sur U; si A est un faisceau d'idéaux fractionnaires, le transporteur B de A dans O est un faisceau d'idéaux fractionnaires dont l'idéal ponctuel en un point x est l'ensemble des fonctions numériques v telles que $vA_x \subset O_x$. Si d est un diviseur, le transporteur de A^d dans O est A^{-d}.

A tout partie fermée $E \neq U$ de la variété U est associé un faisceau A^E d'idéaux fractionnaires de U; les sections de A^E sur un ouvert affine U' sont les fonctions partout définies sur U' qui sont nulles sur $U' \cap E$; si $x \in U$, A^E_x se compose des fonctions définies en x et nulles en tous les points de E appartenant à un voisinage convenable de x. Le faisceau A^E s'appelle le *faisceau de définition de E*.

Soit A un faisceau d'idéaux fractionnaires sur une variété U; si O est le faisceau des anneaux locaux de U, l'ensemble des points $x \in U$ tels que $A_x \subset O_x$ est une partie ouverte non vide de U; soit E le complémentaire de cet ensemble, et soit A^E le faisceau de définition de E. Montrons qu'il y a un exposant $k > 0$ tel que $(A^E)^k A$ soit un faisceau d'idéaux entiers. Soit en effet U' un ouvert affine de U; les sections de A sur U' forment un idéal fractionnaire $\mathfrak{a}$ pour l'algèbre affine $\mathfrak{o}(U')$ de U'; soit $\mathfrak{b}$ l'ensemble des $u \in \mathfrak{o}(U')$ tels que $u\mathfrak{a} \subset \mathfrak{o}(U')$; c'est un idéal (entier) de $\mathfrak{o}(U')$. Si $x \in U'$, l'idéal $\mathfrak{b}\mathfrak{o}(x)$ engendré par $\mathfrak{b}$ dans l'anneau local $\mathfrak{o}(x)$ de x est l'ensemble des $u \in \mathfrak{o}(x)$ tels que $u\mathfrak{a}\mathfrak{o}(x) \subset \mathfrak{o}(x)$, ou encore tels que $uA_x \subset \mathfrak{o}(x)$ (A_x étant l'idéal ponctuel de A en x); on a donc $\mathfrak{b}\mathfrak{o}(x) = \mathfrak{o}(x)$ si $x \notin E$. L'ensemble des zéros (dans U') des fonctions de l'idéal $\mathfrak{b}$ est donc contenu dans E; il en résulte, en vertu du théorème des zéros de Hilbert, que, si $\mathfrak{a}(E)$ est l'idéal des fonctions de $\mathfrak{o}(U')$ nulle sur $U' \cap E$, il y a un $k(U') > 0$ tel que $(\mathfrak{a}(E'))^{k(U')} \subset \mathfrak{b}$. Ceci montre que $(A^E_x)^{k(U')} A_x$ est un idéal entier pour tout $x \in U'$. Il suffit alors de prendre pour k le plus grand des nombres $k(U')$ pour tous les ouverts U' d'un recouvrement ouvert affine fini de U.

Si U est une variété complète et F un faisceau cohérent sur U, les sections de F sur U forment un espace vectoriel de dimension finie sur K ([5]), d'où il résulte que, si A est un faisceau d'idéaux fractionnaires sur U, les fonctions multiples de A forment un espace vectoriel de dimension finie sur K. Nous dirons d'une manière général qu'une variété U est *semi-complète* si, pour tout faisceau A d'idéaux fractionnaires sur U, les fonctions multiples de A forment un espace vectoriel de dimension finie sur K. Une variété complète est évidemment semi-complète. Montrons que, *si U_1 est une variété complète et U une sous-variété ouverte normale de U_1 telle que* $\dim(U - U_1) \leqq \dim U - 2$ (*si $U_1 \neq U$*), *U est semi-complète.* On peut supposer que U_1 est elle-même normale; en effet, il existe une variété normale complète U_2 et un morphisme birationnel f de U_2 sur U_1 tels que (U_2, f) soit un revêtement de U_1. Si $U' = f^{-1}(U)$, et si f' est la restriction de f à U', (U', f') est un revêtement de U; comme f' est birationnel et U normale, f' est un isomorphisme de U' sur U. Supposons donc que U_1 soit normale. Soit A un faisceau d'idéaux fractionnaires sur U. On vient de voir qu'il existe une partie

fermée $E \neq U$ de U et un exposant $k > 0$ tels que $(A^E)^k A$ soit un faisceau d'idéaux entiers. Nous désignerons par E_1 l'adhérence de E dans U_1 et par A^{E_1} le faisceau d'idéaux fractionnaires sur U_1 qui définit l'ensemble E_1; soit enfin A_1 le transporteur de $(A^{E_1})^k$ dans le faisceau des anneaux locaux de U_1. Il suffira d'établir que, si u est une fonction numérique sur U qui est multiple de A, la fonction u_1 sur U_1 qui prolonge u est multiple de A_1, c'est-à-dire que $u_1(A^{E_1}{}_x)^k$ est un idéal entier quel soit $x \in U_1$. Soit S_1 une hypersurface de U_1; montrons que l'ordre de u_1 le long de S_1 est $\geqq -k$ (nous supposons $u_1 \neq 0$). Puisque $\dim(U_1 - U) \leqq \dim U - 2$, S_1 rencontre U; il en résulte tout de suite qu'il existe un point $y \in S_1 \cap U$ tel que toute composante irréductible de E passant par y soit contenue dans S_1. Soit t une fonction numérique définie en y et qui engendre l'idéal premier maximal de l'anneau local de $S_1 \cap U$; il y a alors voisinage U' de U tel que t soit nulle en tout point de $U' \cap E$; il en résulte que $t^k \in (A^E{}_x)^k$, d'où $t^k A_x \in \mathfrak{o}(x)$ et par suite $t^k u \in \mathfrak{o}(x)$ puisque u est multiple de A; on en déduit que l'ordre de u_1 le long de S_1 est $\geqq -k$. Soit maintenant x un point quelconque de U_1; nous voulons montrer que, si $v_1, \cdots, v_k \in A^{E_1}{}_x$, $u_1 v_1 \cdots v_k$ est définie en x. Comme U_1 est normale, il suffit de montrer qu'aucune hypersurface S_1 de U_1 passant par x ne peut être variété de pôles de $u_1 v_1 \cdots v_k$.[1] Supposons d'abord que $S_1 \not\subset E_1$; il y a alors un point $y \in S_1 \cap U$ qui n'appartient pas à E; $A^E{}_x$ est alors l'anneau local de x, d'où il résulte que u est définie en x, donc que u_1 appartient à l'anneau local de S_1; il en est de même de chacune des fonctions v_i, ces fonctions étant définies en x; il en résulte que S_1 n'est pas variété de pôles de $u_1 v_1 \cdots v_k$. Supposons ensuite que $S_1 \subset E_1$; chacune des fonctions v_i, étant nulle sur tous les points de l'intersection avec E_1 d'un voisinage convenable de x, appartient à l'idéal premier maximal $\mathfrak{p}$ de l'anneau local de S_1. On a donc $v_1 \cdots v_k \in \mathfrak{p}^k$; comme l'ordre de u_1 le long de S_1 est $\geqq -k$, $uv_1 \cdots v_k$ appartient à l'anneau local de S_1, et S_1 n'est pas variété de pôles de cette fonction. Notre assertion est donc établie.

On notera qu'il résulte en particulier de là que, si U_1 est une variété complète et normale, l'ensemble U des points simples de U_1 est une variété semi-complète.

Si U est une variété semi-complète, toute fonction numérique u partout définie sur U est constante. En effet, chacune des puissances de u est une

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[1] **Nous appelons zéro d'une fonction numérique u sur une variété U tout point $x \in U$ tel que $(0, x)$ soit adhérent au graphe de u dans $K \times U$, et pôle de u (si $u \neq 0$) tout zéro de u^{-1}. Toute composante irréductible de l'ensemble des zéros (ou des pôles) d'une fonction numérique est une hypersurface ([2], proposition 10, chap. IV, § I). Si U est normale, tout point de U en lequel u n'est pas définie est un pôle de u ([2], proposition 2, chap. V, § I).**

section du faisceau des anneaux locaux; il y a donc un entier $n > 0$ tel que $1, u, \cdots, u^n$ soient linéairement dépendantes sur K, ce qui montre que u est algébrique sur K, d'où $u \in K$.

Si d est un diviseur sur un variété U, l'ensemble des points x tels que $A^d{}_x$ soit distinct de l'anneau local de x est un ensemble fermé $\neq U$, qu'on appelle le *support* de d et qu'on note $\operatorname{Supp} d$. Si u est une fonction de définition de d en tous les points d'une partie ouverte non vide U' de U, $U' \cap \operatorname{Supp} d$ se compose des points $x \in U'$ tels que u et u^{-1} ne soient pas tous deux définis en x. Si d et d' sont des diviseurs, on a $\operatorname{Supp}(d + d') \subset \operatorname{Supp} d \cup \operatorname{Supp} d'$, $\operatorname{Supp}(-d) = \operatorname{Supp} d$.

Soit f un morphisme d'une variété V dans une variété U, et soit d un diviseur sur U dont le support ne contienne pas l'ensemble $f(V)$. Soient y un point de V et u une fonction de définition de d en $x = f(y)$. La fonction u est alors composable avec f, et on a $u \odot f \neq 0$. Soit en effet U' un voisinage ouvert de x tel que u soit fonction de définition de d en tout point de U'. Comme $f(V)$ est un ensemble irréductible non contenu dans $U' \cap \operatorname{Supp} d$, $U' \cap f(V)$ n'est pas contenu dans $U' \cap \operatorname{Supp} d$, ce qui montre qu'il y a un point $x' \in U' \cap f(V)$ tel que u et u^{-1} soient définies en x', ce qui établit notre assertion. Soit $\mathfrak{o}(y)$ l'anneau local de y sur V; on vérifie immédiatement que l'idéal fractionnaire $(u \odot f)\mathfrak{o}(y)$ ne dépend pas de choix de la fonction de définition u de d en x; soit B_y cet idéal fractionnaire. Il est clair que les B_y sont les idéaux ponctuels d'un faisceau d'idéaux fractionnaires principaux sur V; ce dernier définit un diviseur sur V, qu'on appelle *l'image réciproque de d par f*, et qu'on note $f^*(d)$. Les diviseurs d dont les supports ne contiennent pas $f(V)$ forment un sous-groupe du groupe des diviseurs de U, et f^* est un homomorphisme de ce groupe dans le groupe des diviseurs de V. En particulier, si V est une sous-variété de U, f étant l'injection canonique de V dans U, si $f^*(d)$ est défini, on dit que ce diviseur est le *diviseur induit par d sur V*. Si f est un morphisme dominant d'une variété V dans U (i.e. si $f(V)$ est dense dans U), $f^*(d)$ est défini quel que soit le diviseur d. En particulier, si V est une sous-variété ouverte d'une variété U, tout diviseur sur U induit un diviseur sur V. Mais il est important de remarquer que l'application ainsi définie du groupe des diviseurs sur U dans le groupe des diviseurs sur V n'est en général pas surjective; nous verrons cependant qu'elle l'est si U est non singulière.

Soient par ailleurs T une variété et p la projection du produit $T \times U$ sur son second facteur; comme p est surjectif, l'image réciproque par p de tout diviseur d sur U est définie; cette image réciproque sera souvent notée $T \times d$. On a $\operatorname{Supp}(T \times d) = T \times \operatorname{Supp} d$, comme il résulte du fait que, si u est

une fonction numérique sur U, $u \odot p$ n'est définie en un point $(t, x) \in T \times U$ que si u est définie en x. Désignons de plus par t_0 un point de T, par x_0 un point de U, par j l'application $x \to (t_0, x)$ de U dans $T \times U$ et par k l'application $t \to (t, x_0)$ de T dans $T \times U$. Alors $j^*(T \times d)$ est toujours défini et égal à d; $k^*(T \times d)$ n'est défini que si x_0 n'appartient pas à $\operatorname{Supp} d$, et est alors nul. Ces faits résultent immédiatement des définitions.

Soient f un morphisme d'une variété V dans une variété U et g un morphisme d'une variété W dans V; soit d un diviseur sur U. Si $(f \circ g)^*(d)$ est défini, il en est de même de $f^*(d)$ et de $g^*(f^*(d))$, et on a

$$g^*(f^*(d)) = (f \circ g)^*(d).$$

Soit d un diviseur sur une variété *normale* U, et soit S une hypersurface de U. On sait que l'anneau local $\mathfrak{o}(S)$ de S sur U est un anneau local principal. Si $\mathfrak{p}_S$ est l'idéal premier maximal de cet anneau, les idéaux fractionnaires pour $\mathfrak{o}(S)$ sont les puissances d'exposants de signes quelconques de $\mathfrak{p}_S$. En particulier, on a, si $x \in S$, $A^d{}_x\mathfrak{o}(S) = \mathfrak{p}_S{}^{k(S)}$, $k(S)$ étant un entier dont on voit tout de suite qu'il ne dépend que de d et de S, non du choix de x sur S. Il est clair que $k(S) = 0$ si S n'est pas contenu dans $\operatorname{Supp} d$; il n'y a donc qu'un nombre fini d'hypersurfaces S pour lesquelles $k(S) \neq 0$, et on peut associer à d le cycle $Z(d) = \sum_S k(S) S$ de codimension 1 sur U. On obtient ainsi un homomorphisme $d \to Z(d)$ du groupe des diviseurs dans le groupe des cycles de codimension 1. Nous allons montrer que cet homomorphisme est *injectif*. Il suffira pour cela d'établir que $\operatorname{Supp} d$ *est la réunion des hypersurfaces* S *pour lesquelles* $k(S) \neq 0$. On sait déjà que ces hypersurfaces sont contenues dans $\operatorname{Supp} d$. Soient x un point de $\operatorname{Supp} d$, et u une fonction de définition de d en x. L'une au moins des fonctions u, u^{-1} n'est pas définie en x. Comme U est normale, il en résulte que x est un zéro ou un pôle de u. Si par exemple x est un pôle de u, il passe par x une composante irréductible S de l'ensemble des pôles de u, et on sait que S est une hypersurface; comme u n'est définie en aucun point de S, on a $k(S) < 0$. On voit de même que, si x est un zéro de u, il passe par x une hypersurface S pour laquelle $k(S) > 0$.

On notera que le raisonnement qu'on vient de faire prouve que, si d n'est pas positif en un point x, il passe par x au moins une hypersurface S pour laquelle $k(S) < 0$.

Il est important de remarquer que l'application $d \to Z(d)$ n'est en général pas surjective; par exemple, si U est un cône quadratique, le cycle constitué par une génératrice du cône, prise avec le coefficient 1, n'est le cycle associé à aucun diviseur. On a cependant le résultat suivant:

PROPOSITION 1. *Si U est une variété non singulière, tout cycle de codimension* 1 *sur U est associé à un diviseur sur U.*

Il suffit de montrer que, si S est une hypersurface, le cycle $1 \cdot S$ est associé à un diviseur. Soit A^S le faisceau d'idéaux qui définit S; comme U est non singulière, il est connu que, pour tout $x \in S$, l'idéal de définition de S en x est principal; A^S est donc un faisceau d'idéaux principaux, et est par suite associé à un diviseur d; il est clair que $Z(d) = 1 \cdot S$.

COROLLAIRE. *Soient U une variété non singulière, V une sous-variété ouverte de U et i l'injection canonique de V dans U; i^* est alors une application surjective du groupe des diviseurs de U sur celui de V.*

En effet, si S_V est une hypersurface de V, son adhérence S dans U est une hypersurface de U; si d est le diviseur sur U auquel S est associée, le cycle sur V associé à $i^*(d)$ est S_V.

PROPOSITION 2. *Soit h un morphisme surjectif propre d'une variété X' dans une variété X; supposons que, pour tout $x' \in X'$, toute fonction numérique u sur X telle que $u \odot h$ soit définie en x' soit définie au point $h(x')$. Soit d' un diviseur sur X'; supposons que, pour tout $x \in X$, il existe une fonction numérique v sur X telle que $v \odot h$ soit fonction de définition de d' en tout point de $h^{-1}(x)$. Il existe alors un diviseur d et un seul sur X tel que $d' = h^*(d)$; on a* $\operatorname{Supp} d' = h^{-1}(\operatorname{Supp} d)$.

Si $x \in X$, soit $\mathfrak{o}(x)$ l'anneau local de x. Soit v une fonction numérique sur X tell que $h \odot v$ soit fonction de définition de d' en tout point de $h^{-1}(x)$; l'idéal fractionnaire $\mathfrak{o}(x)v$ ne dépend alors que de x et d'; en effet, si v_1 est une autre fonction qui possède la même propriété que v, $h \odot (v^{-1}v\ )$ est définie en tout point de $h^{-1}(x)$ et y prend une valeur $\neq 0$, ce qui implique que $v^{-1}v_1$ est définie en x et y prend une valeur $\neq 0$, donc que $\mathfrak{o}(x)v = \mathfrak{o}(x)v_1$; posons $A_x = \mathfrak{o}(x)v$. Montrons qu'il y a un voisinage de x tel que l'on ait $A_y = \mathfrak{o}(y)v$ pour tout point de ce voisinage. L'ensemble X'_0 des points $x' \in X'$ tels que $h \odot v$ soit fonction de définition de d' en x' est ouvert; comme h est propre, $h(X' - X'_0)$ est fermé; de plus, cet ensemble ne contient pas x; $X_0 = X - h(X' - X'_0)$ est donc un voisinage de x, et il est clair que $A_y = \mathfrak{o}(y)v$ pour tout $y \in X_0$. Il résulte de là que les $A_x (x \in X)$ sont les idéaux ponctuels d'un faisceau d'idéaux fractionnaires principaux sur X; ce faisceau définit un diviseur d; il est clair que $d' = h^*(d)$ et que d est le seul diviseur sur X possédant cette propriété. Les notations étant comme ci-dessus, supposons que $x \in \operatorname{Supp} d$; alors l'une au moins des fonctions v, v^{-1} n'est pas définie en x. Il en résulte que, pour tout $x' \in h^{-1}(x)$, l'une au moins

des fonctions $h \odot v$, $(h \odot v)^{-1}$ n'est pas définie en x', ce qui montre que $h^{-1}(x) \subset \operatorname{Supp} d'$. Comme on sait déjà que $\operatorname{Supp} d'$ est contenu dans $h^{-1}(\operatorname{Supp} d)$, on a $\operatorname{Supp} d' = h^{-1}(\operatorname{Supp} d)$.

Remarque. Supposons que (X', h) soit un revêtement de X. La condition que nous avons imposée à h dans l'énoncé de la Proposition 2 est alors satisfaite dans le cas où X est normale ([2], chap. V, § V, proposition 4). Elle l'est également si on suppose que h est un revêtement galoisien non ramifié ([3], chap. V, § 11, Corollaire 2 à la Proposition 7) ; rappelons que cela signifie qu'il existe un groupe G d'automorphismes de X' tel que les orbites relativement à G des points de X' soient exactement les ensembles $h^{-1}(x)$, $x \in X$, et que de plus h n'est ramifié en aucun point de $h^{-1}(x)$, ce qui peut se traduire par la condition que G opère sans point fixe sur X'.

Nous utiliserons dans la suite le lemme suivant:

LEMME 1. *Soient d un diviseur sur une variété U et $x_1, \cdots, x_m$ un nombre fini de points de U qui appartiennent à un même morceau affine de U; il y a alors une fonction numérique sur U qui est fonction de définition de d en chacun des points x_i.*

On peut supposer les points x_i mutuellement distincts. Soit u_i une fonction de définition de d en x_i; on a donc $\operatorname{div} u_i = d + d_i$, où d_i est un diviseur dont le support ne contient pas x_i (nous notons $\operatorname{div} u$ le diviseur principal associé à une fonction $u \neq 0$). Soit J_i le faisceau d'idéaux associé à l'ensemble $\operatorname{Supp} d_i$; il résulte de ce qui a été dit plus haut qu'il existe un entier $k_i \geqq 0$ tel que $J_i^{k_i} A^{d_i}$ soit un faisceau d'idéaux entiers. Soit U' un morceau affine de U contenant les points x_i; comme $x_i \notin \operatorname{Supp} d_i$, il y a une fonction numérique z_i partout définie sur U' qui est nulle sur $U' \cap \operatorname{Supp} d_i$, qui prend la valeur 0 en tous les points x_j, $j \neq i$, mais qui ne prend pas la valeur 0 en x_i. Posons $u_i' = u_i z_i^{k_i+1}$; u_i' est encore fonction de définition de d en x_i; si $\operatorname{div} u_i' = d + d_i'$, d_i' est la somme de $\operatorname{div} z_i$ et d'un diviseur qui est $\geqq 0$ en tout point de U', ce qui montre que $x_j \in \operatorname{Supp} d_i'$ si $j \neq i$; par contre, x_i n'est pas dans $\operatorname{Supp} d_i'$. Soit $u = \sum_{i=1}^{m} u_i'$; on a alors $u u_i'^{-1} = 1 + \sum_{j \neq i} u_j' u_i'^{-1}$; si $j \neq i$, on a $\operatorname{div} u_j' u_i'^{-1} = d_j' - d_i'$; or d_j' est positif en x_i et $x_i \notin \operatorname{Supp} d_i'$; il en résulte que $u_j' u_i'^{-1}$ est définie en x_i; comme $x_i \in \operatorname{Supp} d_j'$, on a $(u_j' u_i'^{-1})(x_i) = 0$. Il résulte de là que $u u_i'^{-1}$ est définie en x_i et y prend la valeur 1, donc que u est fonction de définition de d en x_i. Ceci étant vrai pour tout i, le lemme est établi.

PROPOSITION 3. *Soit (X', h) un revêtement galoisien non ramifié d'une*

variété X; *supposons que, pour tout* $x \in X$, $h^{-1}(x)$ *soit contenu dans un morceau affine de* X'. *Soit* d' *un diviseur sur* X' *tel que l'on ait* $s^*(d') = d'$ *pour tout automorphisme* s *du revêtement* (X', h); *il existe alors un diviseur* d *et un seul sur* X *tel que* $d' = h^*(d)$; *on a* $\operatorname{Supp} d' = h^{-1}(\operatorname{Supp} d)$.

Tenant compte de la Proposition 2 et de la remarque qui suit la démonstartion de cette proposition,[2] on voit qu'il suffit de montrer que, si $x \in X$, il y a une fonction numérique v sur X telle que $v \odot h$ soit fonction de définition de d' en tout point de $h^{-1}(x)$. Il existe une fonction numérique v' sur X' qui est fonction de définition de d' en tout point de $h^{-1}(x)$ (Lemme 1). Il est clair que, si G est le groupe des automorphismes du revêtement (X', h), toute fonction de la forme $v' \odot s$ $(s \in G)$ possède la même propriété que v'; si donc $x' \in h^{-1}(x)$, $v'^{-1}(v' \odot s)$ est définie et prend une valeur $a_s \neq 0$ en x'. Puisque h est non ramifié, on a $s(x') \neq x'$ pour toute opération s distincte de l'identité de G. Comme $h^{-1}(x)$ est contenu dans un morceau affine de X', il existe une fonction numérique z' sur X', définie en tout point de $h^{-1}(x)$, telle que $z'(x') = 1$, $z'(s(x')) = 0$ pour tout $s \in G$ distinct de l'identité. Posons $v'_1 = \sum_{s \in G} (z'v' \odot s)$; $v'^{-1}v'_1$ est définie en x' et y prend la valeur $a_e \neq 0$, ce qui signifie que v'_1 est fonction de définition de d' en x'. Or on a $v' \odot s = v'_1$ pour tout $s \in G$; il en résulte d'abord que v'_1 est fonction de définition de d' en tout point de $h^{-1}(x)$, puis (le revêtement h étant galoisien, donc séperable) que v'_1 se met sous la form $v \odot h$, v étant une fonction numérique sur X. La Proposition 3 est donc étable.

Remarque. La condition que, pour tout $x \in X$, $h^{-1}(x)$ soit contenu dans un morceau affine de X' est satisfaite si X' est normale; soit alors en effet X_0 un morecau affine de X contenant x; posons $X'_0 = h^{-1}(X_0)$, et désignons par h_0 la restriction de h à X'_0; (X'_0, h_0) est alors un revêtement normal de la variété affine X_0, d'où il résulte que X'_0 est affine. On peut montrer que la condition en question est satisfaite pour tout revêtement; mais nous n'aurons pas besoin de ce résultat plus fin.

II. Familles algebriques de diviseurs. Soient T et U des variétés. Pour tout $t \in T$, nous désignerons par j_t l'application $x \to (t, x)$ de U dans $T \times U$.

DÉFINITION 1. *Une application* f *de* T *dans le groupe des diviseurs de* U *s'appelle une famille algébrique de diviseurs de* U (*paramétrée par* T)

[2] **Il suffira d'ailleurs pour la suite de savoir que la Proposition 3 est vraie dans le cas où X est normale.**

s'il existe un diviseur D de $T \times U$ tel que, pour tout $t \in T$, $j_t^*(D)$ soit défini et égal à $f(t)$. On dit alors que D est un diviseur de définition de la famille f.

Il est clair que les familles de diviseurs de U paramétrées par T forment un groupe additif.

PROPOSITION 1. *Soit f une famille algébrique de diviseurs de U paramétrée par T; soit h un morphisme d'une variété T' dans la variété T. Alors $f \circ h$ est une famille algébrique de diviseurs de U paramétrée par T'; si D est un diviseur de définition de $f, h^*(D)$ est défini et est un diviseur de définition de $f \circ h$.*

Dans l'énoncé précédent, ainsi qu'en plusieurs endroits de la suite de ce mémoire, nous faisons la convention de notation suivante: si h est un morphisme de T' dans T, nous désignons encore par h le morphisme $(t', x) \to (h(t'), x)$ de $T' \times U$ dans $T \times U$.

Si $t' \in T'$, soit $j'_{t'}$ l'application $x \to (t', x)$ de U dans $T' \times U$; on a $j_{h(t')} = h \circ j'_{t'}$, et $j_{h(t')}^*(D)$ est défini et égal à $f(h(t'))$; on en conclut que $h^*(D)$ est défini, qu'il en est de même de $(j'_{t'})^*(h^*(D))$, et que ce dernier diviseur est égal à $j_{h(t')}^*(D) = f(h(t'))$, ce qui démontre la Proposition 1.

PROPOSITION 2. *Soit f une famille algébrique de diviseurs de U paramétrée par T; soit g un morphisme dominant d'une variété U' dans U. Alors $t \to g^*(f(t))$ est une famille algébrique de diviseurs de U'; si D est un diviseur de définition de $f, g^*(D)$ est un diviseur de définition de la famille $t \to g^*(f(t))$.*

On notera que $g^*(D)$ est défini puisque l'application $(t, x') \to (t, g(x'))$ est un morphisme dominant de $T \times U'$ dans $T \times U$. Soit t un point de T; il existe un point $x \in U$ tel que $(t, x) \notin \operatorname{Supp} D$; de plus, les points qui possèdent cette propriété forment une partie ouverte de U, qui rencontre donc $g(U')$; il y a donc un point $x' \in U'$ tel que $(t, g(x')) \notin \operatorname{Supp} D$, d'où $(t, x') \notin \operatorname{Supp} g^*(D)$. Il en résulte que, si on désigne par j'_t l'application $x' \to (t, x')$ de U' dans $T \times U'$, $(j'_{t'})^*(g^*(D))$ est défini; de plus, $(g \circ j'_t)^*(D)$ est défini. Comme $g \circ j'_t = j_t \circ g$, il en résulte que

$$(j'_t)^*(g^*(D)) = (g \circ j'_t)^*(D) = g^*(j_t^*(D)) = g^*(f(t)),$$

ce qui démontre la Proposition 2.

THÉORÈME 1. *Soient T et U des variétés et f une famille algébrique de diviseurs de U paramétrée par T; soit D un diviseur de définition de f. Si on a $f(t) \geqq 0$* (resp. $f(t) = 0$) *pour tous les points t d'une partie dense de T,*

on a $D \geqq 0$ (resp. $D = 0$), *et par suite* $f(t) \geqq 0$ (resp. $f(t) = 0$) *pour tous les points de* T.

Les assertions relatives au cas où $f(t) = 0$ pour tous les points d'une partie dense de T se déduisent de celles relatives au cas où $f(t) \geqq 0$ pour tous les points d'une partie dense en observant qu'une condition nécessaire et suffisante pour qu'un diviseur d sur une variété soit nul est que l'on ait à la fois $d \geqq 0$ et $-d \geqq 0$. Il nous suffira donc de prouver les premières de ces assertions. Montrons d'abord qu'on peut se ramener au cas où T et U sont affines. Soit (t_0, x_0) un point de $T \times U$; on veut montrer que D est positif en ce point. Soient T_0 et U_0 des morceaux affines de T et U contenant t_0 et x_0 respectivement; soit i l'application canonique de U_0 dans U. La restriction de l'application $t \to i^*(f(t))$ à T_0 est la famille de diviseurs de U_0 définie par le diviseur D_0 induit par D sur $T_0 \times U_0$. L'ensemble des points $t \in T_0$ tels que $i^*(f(t)) \geqq 0$ est dense; si le théorème est prouvé pour les variétés affines, il en résultera que $D_0 \geqq 0$, donc que D est positif en (t_0, x_0).

Supposant T et U affines, montrons qu'on peut se ramener au cas où D est principal. Soit (t_0, x_0) un point de $T \times U$, et soit w une fonction de définition de D en ce point; on a donc $\operatorname{div} w = D + D'$, où D' est un diviseur dont le support ne contient pas (t_0, x_0). Puisque $T \times U$ est une variété affine, il y a une fonction polynome z sur cette variété qui est nulle sur $\operatorname{Supp} D'$ mais qui prend une valeur $\neq 0$ en (t_0, x_0). On voit alors facilement qu'il y a un exposant $k \geqq 0$ tel que $\operatorname{div} z^k + D' \geqq 0$ (cf. § I). Soit $w' = wz^k$; w' est encore une fonction de définition de D en (t_0, x_0), et on a $\operatorname{div} w' = D + D''$, où D'' est un diviseur $\geqq 0$. Il y a un voisinage affine T_1 de t_0 dans T tel que $T_1 \times \{x_0\}$ ne rencontre pas $\operatorname{Supp} D''$; si $t \in T_1$, il résulte du fait que $j_t^*(D) = f(t)$ est défini qu'il en est de même de $j_t^*(\operatorname{div} w)$; de plus, $j_t^*(\operatorname{div} w) = f(t) + j_t^*(D'')$, d'où il résulte que $j_t^*(\operatorname{div} w)$ est $\geqq 0$ pour tous les points d'une partie dense de T_1; si on peut en conclure que $\operatorname{div} w \geqq 0$, il en résultera que D est positif en (t_0, x_0).

Supposons à partir de maintenant que T et U soient affines et que $D = \operatorname{div} w$ soit principal. Nous allons montrer qu'on peut se ramener au cas où on suppose de plus que T et U sont normales. Supposons le théorème établi dans ce cas. Il existe des variétés normales T' et U' et des morphismes $h : T' \to T$, $g : U' \to U$ tels que h et g soient des morphismes de revêtement; il est bien connu que T' et U' sont alors encore des variétés affines. Soit r le morphisme $(t', x') \to (h(t'), g(x'))$ de $T' \times U'$ dans $T \times U$; il résulte des Propositions 1 et 2 que l'application $f' : t' \to g^*(h(t'))$ est une famille algébrique de diviseurs de U' paramétrée par T' qui admet $r^*(D)$ comme diviseur

de définition. Si A est l'ensemble des points $t \in T$ tels que $f(t) \geqq 0$, on a $f'(t') \geqq 0$ toutes les fois que $t' \in h^{-1}(A)$; comme A est dense dans T, $h^{-1}(A)$ est dense dans T'. En vertu de l'hypothèse faite, il en résulte que le diviseur principal $r^*(D)$ est $\geqq 0$. Si $D = \operatorname{div} w$, on a $r^*(D) = \operatorname{div}(w \odot r)$; $w \odot r$ est donc une fonction numérique partout définie sur $T' \times U'$. Observons par ailleurs que, si $t \in A$, $f(t) = j_t^*(\operatorname{div} w) = \operatorname{div} w \odot j_t$ est partout définie sur U. On va montrer que l'espace vectoriel V engendré par les fonctions $w \odot j_t$ pour tous les $t \in A$ est de dimension finie. Soit t un point de A, et soit t' un point de $h^{-1}(t)$. On a $j_t \circ g = r \circ j'_{t'}$, où $j'_{t'}$ est l'application $x' \to (t', x')$ de U' dans $T' \times U'$; il en résulte que $w \odot j_t \odot g = (w \odot r) \odot j'_{t'}$. Par ailleurs, l'application $u \to u \odot g$ est un isomorphisme du corps des fonctions numériques sur U sur un sous-corps du corps des fonctions numériques sur U'. Pour montrer que V est de dimension finie, il suffira donc de montrer que l'espace vectoriel engendré par les fonctions $(w \odot r) \odot j'_{t'}$ (pour tous les points $t' \in T'$) est de dimension finie dans le corps des fonctions sur U'. Mais, comme $w \odot r$ est partout définie sur $T' \times U'$, il y a des fonctions θ'_i $(1 \leqq i \leqq h)$ partout définies sur T' et u'_i partout définies sur U' telles que $(w \odot r)(t', x') = \sum_{i=1}^{h} \theta'_i(t') u'_i(x')$ pour tout $(t', x') \in T' \times U'$; $(w \odot r) \odot j'_{t'}$ est donc toujours une combinaison linéaire de $u'_1, \cdots, u'_h$. Ceci étant, soit $(u_1, \cdots, u_m)$ une base de V; si $t \in A$, il y a des éléments $\theta_i(t)$ $(1 \leqq i \leqq m)$ de K tels que $w \odot j_t = \sum_{i=1}^{m} \theta_i(t) u_i$. Il s'agit de montrer que les applications θ_i peuvent se prolonger en des fonctions numériques partout définies sur T. Soit t_1 un point de T; comme $j_{t_1}^*(\operatorname{div} w)$ est défini, l'ensemble U_1 des points $x \in U$ tels que w soit définie en (t_1, x) est ouvert et non vide. Comme $u_1, \cdots, u_m$ sont linéairement indépendantes, il est facile de voir qu'il existe des points $x_1, \cdots, x_m$ de U_1 tels que $\det(u_i(x_j)) \neq 0$. Comme w est définie aux points (t_1, x_j), il y a un voisinage ouvert T_1 de t_1 tel que w soit définie en tout point de chacun des ensembles $T_1 \times \{x_j\}$. Soit t un point de $T_1 \cap A$; alors les $\theta_i(t)$ peuvent s'obtenir par la résolution du système d'équations linéaires $\sum_{i=1}^{m} \theta_i(i) u_i(x_j) = w(t, x_j)$ $(1 \leqq j \leqq m)$. Or les application $t \to w(t, x_j)$ $(t \in T)$ sont des fonctions numériques partout définies sur T_1; il en résulte aussitôt que les restrictions des θ_i à $A \cap T_1$ peuvent se prolonger en des fonctions numériques partout définies sur T_1. Soient maintenant T_1 et T_2 des parties ouvertes non vides de T tells que les restrictions des θ_i à $T_1 \cap A$ (resp. $T_2 \cap A$) puissent se prolonger en des fonctions numériques $\theta_{i;1}$ (resp. $\theta_{i;2}$) sur T_1 (resp. T_2). Alors, pour chaque i, $\theta_{i;1}$ coincide avec $\theta_{i;2}$ sur

l'ensemble $A \cap T_1 \cap T_2$, qui est dense dans $T_1 \cap T_2$; il en résulte que $\theta_{i;1}$ coincide avev $\theta_{i;2}$ sur $T_1 \cap T_2$. Il résulte tout de suite de là que les applications θ_i peuvent, d'une manière et d'une seule, se prolonger en des fonctions numériques partout définies sur T, que nous désignerons encore par θ_i. Soit alors w_0 la fonction numérique partout définie sur $T \times U$ donnée par la formule $w_0(t,x) = \sum_{i=1}^{m} \theta_i(t) u_i(x)$; si W est l'ensemble de définition de w, $w - w_0$ prend la valeur 0 en tout point de l'ensemble $W \cap (A \times U)$, qui est dense dans W; elle est donc nulle, ce qui montre que $w = w_0$, donc que w est partout définie et par suite que $\operatorname{div} w \geqq 0$.

Il nous reste à démontrer le théorème dans le cas où T et U sont normales et où $D = \operatorname{div} w$ est un diviseur principal. Pour montrer que w est partout définie sur $T \times U$, il suffira, puisque $T \times U$ est normale, de montrer que w n'a pas de pôle. Or, il est bien connu que, si w avait au moins une variété de pôles, soit S, il existerait un point $(t,x) \in S$ tel que w^{-1} soit définie et prenne la valuer 0 en (t,x); mais alors la fonction $(w \odot j_t)^{-1}$ serait définie et prendrait la valuer 0 en x, de sorte que x serait un pôle de $w \odot j_t$ et que $f(t) = \operatorname{div} w \odot j_t$ ne serait pas un diviseur $\geqq 0$. Le Théorème 1 est donc établi.

COROLLAIRE 1. *Si f est une famille algébrique de diviseurs d'une variété U paramétrée par une variété T, il n'y a qu'un seul diviseur sur $T \times U$ qui soit diviseur de définition de la famille f.*

Cela résulte immédiatement du Théorème 1.

COROLLAIRE 2. *Soit f une application d'une variété T dans l'ensemble des diviseurs d'une variété U. Supposons que chaque point de T ait un voisinage ouvert T' tel que la restriction de f à T' soit une famille algébrique paramétrée par T'. L'application f est alors une famille algébrique de diviseurs.*

Il existe un recouvrement $(T_i)_{i \in I}$ de T par des ensembles ouverts non vides tels que, pour tout i, la restriction de f à T_i soit une famille algébrique paramétrée par T_i; soit D_i le diviseur de définition de cette famille; c'est un diviseur de $T_i \times U$. Si $i, j \in I$, les diviseurs induits par D_i et D_j sur $(T_i \cap T_j) \times U$ définissent la même famille algébrique de diviseurs, et sont par suite égaux. Il en résulte immédiatement qu'il existe un diviseur D sur $T \times U$ tel que, pour tout i, D_i soit le diviseur induit par D sur $T_i \times U$. Il est clair que l'on a $j_t^*(D) = f(t)$ pour tout $t \in T$, ce qui démontre le Corollaire 2.

COROLLAIRE 3. *Soit f une famille algébrique de diviseurs d'une variété*

U paramétrée par une variété T. L'ensemble des points $t \in T$ tels que $f(t) \geqq 0$ est fermé, et il en est de même de l'ensemble des points t tels que $f(t) = 0$.

Comme dans la démonstration du Théorème 1, il suffit de démontrer que l'ensemble E des points t tels que $f(t) \geqq 0$ est fermé. Soient E_1 une composante irréductible de E et T' son adhérence; la restriction de f à T' est une famille algébrique paramétrée par T' et qui fait correspondre des diviseurs $\geqq 0$ aux points de la partie dense E_1 *de* T'; elle est donc positive, d'où $T' \subset E$. Ceci étant vrai pour toute composante irréductible de E, E est fermé.

Corollaire 4. *Soient f et f' des familles algébriques de diviseurs d'une même variété U paramétrées par des variétés T et T'. L'ensemble des points $(t, t') \in T \times T'$ tels que $f(t) = f'(t')$ est alors fermé.*

Cela résulte du Corollaire 3 et du fait que l'application $(t, t') \rightarrow f(t) - f'(t')$ est une famille algébrique paramétrée par $T \times T'$ (Proposition 1).

Nous allons maintenant donner deux exemples importants de familles algébriques de diviseurs.

Soient U une variété et V un sous-espace vectoriel de dimension finie > 0 du corps $F(U)$ des fonctions numériques sur U. Puisque V est de dimension finie, l'ensemble U_0 des points de U en lesquels toutes les fonctions de V sont définies est ouvert et non vide. L'application $(u, x) \rightarrow u(x)$ de $V \times U_0$ dans K se prolonge en une fonction numérique w sur $V \times U$. Soit V_0 l'ensemble des éléments $\neq 0$ de V; si $u \in V$, soit j_u l'application $x \rightarrow (u, x)$ de U dans $V \times U$; il est clair que w est toujours composable avec j_u, et que $w \odot j_u = u \neq 0$ si $u \in V_0$. Soit w_0 la restriction de w à $V_0 \times U$; il résulte de ce qu'on vient de dire que $\operatorname{div} w_0$ est le diviseur de définition d'une famille algébrique de diviseurs de U paramétrée par V_0, qui n'est autre que l'application $u \rightarrow \operatorname{div} u \, (u \in V_0)$.

Désignons maintenant par $\mathfrak{P}(V)$ l'espace projectif associé à V, qui se compose des sous-espace de dimension 1 de V, et par φ l'application canonique $u \rightarrow Ku$ de V_0 sur $\mathfrak{P}(V)$; comme $\operatorname{div} cu = \operatorname{div} u$ si c est un élément $\neq 0$ de K, on voit que $\operatorname{div} u$ ne dépend que du point $\zeta = \varphi(u)$. Nous poserons $\operatorname{div} u = \operatorname{div} \zeta$ si $\zeta = \varphi(u)$. Montrons que l'application $\zeta \rightarrow \operatorname{div} \zeta$ est une famille algébrique de diviseurs de U paramétrée par $\mathfrak{P}(V)$. Soit ζ_0 un point de $\mathfrak{P}(V)$. Il existe alors un voisinage ouvert T_0 de ζ_0 dans $\mathfrak{P}(V)$ et un morphisme r de T_0 dans V_0 tels que $\varphi \circ r$ soit l'application indentique de T_0; si $\zeta \in T_0$, on a $\operatorname{div} \zeta = \operatorname{div} r(\zeta)$, d'où il résulte que la restriction à T_0 de l'application $\zeta \rightarrow \operatorname{div} \zeta$ est une famille algébrique de diviseurs de T_0; on conclut alors au moyen du Corollaire 2 au Théorème 1. L'application $\zeta \rightarrow \operatorname{div} \zeta$ s'appelle le *système linéaire de diviseurs de U défini par V.*[3] Si U est une variété semi-

complète, l'application $\zeta \rightarrow \operatorname{div} \zeta$ est injective, car toute fonction numérique de diviseur nul sur U est alors constante.

Soit maintenant C une courbe normale. Soit r un entier $\geqq 0$; soit C^r le produit de r exemplaires de C. Les permutations des facteurs du produit C^r définissent un groupe fini P d'automorphismes de C^r. Il est bien connu que C est une variété quasi-projective; il en est donc de même de C^r, de sorte que toute partie affine de C^r est contenue dans un morceau affine de cette variété. Il existe donc une variété quotient S^r de C^r par P: il existe un morphisme s_r de C^r sur S^r tel que (C^r, s_r) soit un revêtement galoisien de groupe P de S^r, et S^r est normale. On dit que S^r est la *puissance symétrique* r-ième de C, et s_r le *morphisme canonique* $C^r \rightarrow S^r$.

Soit maintenant f une famille algébrique de diviseurs d'une variété normale U paramétrée par C. Il résulte immédiatement de la Proposition 1 que l'application

$$(x_1, \cdots, x_r) \rightarrow \sum_{i=1}^{r} f(x_i)$$

est une famille algébrique m de diviseurs de U paramétrée par C^r. Nous allons montrer que m peut se mettre sous la fome $g \circ s_r$, où g est une famille algébrique de diviseurs de U paramétrée par S_r. Soit M le diviseur de définition de m. Soient $(a_1, \cdots, a_r)$ un point de C^r, b un point de U et w une fonction numérique sur $C \times U$ qui est fonction de définition de M en chacun des points (a_i, b) (Lemme 1, § I; on notera que, C étant une variété quasi-projective, toute partie finie de C est contenue dans un morceau affine). Soient $q_1, \cdots, q_r$ les projections de C^r sur ses divers facteurs; nous désignons encore par q_i l'application $((x_1, \cdots, x_r), y) \rightarrow (x_i, y)$ de $C^r \times U$ sur $C \times U$. Il est clair que $\prod_{i=1}^{r} (w \odot q_i)$ est une fonction de définition de M en $((a_1, \cdots, a_r), b)$; soit z cette fonction. Il est clair que, pour toute opération p du groupe P, on a $z \odot p = z$ (p étant identifié à l'application $((x_1, \cdots, x_r), y) \rightarrow (p(x_1, \cdots, x_r), y)$). On en conclut que z peut se mettre sous la forme $v \odot s_r$ (s_r étant identifié à l'application $((x_1, \cdots, x_r), y) \rightarrow (s_r(x_1, \cdots, x_r), y)$), v étant une fonction numérique sur $S^r \times U$. De plus, il est également clair que z est aussi fonction de définition de M en tout point de la forme $(p(a_1, \cdots, a_r), b)$, $p \in P$. Il résulte alors de la Proposition 2, § I que M se met sous la forme $s_r^*(D)$, D étant un diviseur sur $S^r \times U$, et que $\operatorname{Supp} M = s_r^{-1}(\operatorname{Supp} D)$; il résulte de cette dernière égalité que l'ensemble

[3] On reserve d'habitude le nom de système linéaire aux applications de la forme $\zeta \rightarrow \operatorname{div} \zeta + d_0$, où d_0 est un diviseur tel que $\operatorname{div} \zeta + d_0$ soit positif pour tout ζ. Il nous a semblé plus commode d'utiliser la terminologie donnée dans le texte.

$\{s_r(x_1,\cdots,x_r)\}\times U$ (où $(x_1,\cdots,x_r)\in C^r$) n'est jamais contenu dans Supp D, de sorte que D définit une famille algébrique g de diviseurs de U paramétrée par S^r; il est clair que $m = g\circ s_r$.

On peut appliquer ce qui précède au cas où f est la famille de diviseurs de C paramétrée par C qui associe à tout $x\in C$ le diviseur, noté 1.x, auquel est associé le cycle 1.x; le diviseur de définition de cette famille est le diviseur de $C\times C$ auquel est associé le cycle constitué par la diagonale de $C\times C$ prise avec le coefficient 1. On voit donc qu'il existe une famille algébrique d_r de diviseurs de C paramétrée par S^r telle que

$$d_r(s_r(x_1,\cdots,x_r)) = x_1+\cdots+x_r$$

pour tout $(x_1,\cdots,x_r)\in C^r$; nous dirons que d_r est la *famille canonique* de diviseurs de C paramétrée par S^r.

Si m est un diviseur quelconque de C, le cycle associé à m se met sous la form $\sum_{i=1}^{k} a_i x_i$, $x_1,\cdots,x_k$ étant des points mutuellement distincts de C; le nombre $\sum_{i=1}^{k} a_i$ s'appelle le *degré* de m; de plus, dans la démonstration qui va suivre, nous appellerons hauteur de m le nombre $\sum_{i=1}^{k}|a_i|$. Rappelons que si C est complète, tout diviseur principal sur C est de degré 0.

Proposition 3. *Soit f une famille algébrique de diviseurs d'une courbe complète normale C paramétrée par une variété T; le degré de $f(t)$ est alors indépendant de t.*

Pour tout $r\geqq 0$, soit d_r la famille canonique de diviseurs de C paramétrée par S^r. Il est clair que, si m est un diviseur quelconque sur C, il y a des entiers $r\geqq 0$, $r'\geqq 0$ et des points $z\in S^r$, $z'\in S^{r'}$ tels que $m = d_r(z) - d_{r'}(z')$; on peut de plus supposer que $r+r'$ est égal à la hauteur de m, que nous noterons $h(m)$. Soient r, r' des entiers $\geqq 0$ quelconque; l'ensemble des points $(z,z',t)\in S^r\times S^{r'}\times T$ tels que $f(t)+d_{r'}(z') = d_r(z)$ est fermé (cf. Corollaire 4 au Théorème 1); l'image $H_{r,r'}$ de cet ensemble par la projection de $S^r\times S^{r'}\times T$ sur T est donc fermée ($S^r\times S^{r'}$ étant une variété complète). La variété T est la réunion des $H_{r,r'}$ pour tous les couples d'entiers $r\geqq 0$, $r'\geqq 0$; de plus, si les hauteurs des diviseurs $f(t)$, $t\in T$, restent bornées, au moins pour les points d'une partie dense de T, il y aura un nombre h tel que la réunion des $H_{r,r'}$ pour $r+r'\leqq h$ soit dense dans T, donc soit T tout entier; comme T est irréductible, il en résultera qu'il y a un couple (r,r') tel que $H_{r,r'} = T$, et $f(t)$ sera toujours de degré $r-r'$. Nous sommes donc ramenés à prouver que les hauteurs des diviseurs $f(t)$ restent bornées quand t parcourt les points d'une partie dense convenable de T.

Soit w une fonction numérique $\neq 0$ sur $T \times C$. Nous allons montrer qu'il y a une partie ouverte non vide T_1 de T et un entier h_1 tels que, pour $t \in T_1$, $j_t^*(\operatorname{div} w)$ soit défini et de hauteur $\leqq h_1$. Soient p et q les projections de $T \times C$ sur son premier et son second facteur; il y a alors des fonctions numériques $u_1, \cdots, u_n$ linéairement indépendantes sur C et des fonctions numériques θ_i, θ_i' sur T $(1 \leqq i \leqq n)$ telles que

$$w = (\sum_{i=1}^{n} (\theta_i \odot p)(u_i \odot q))(\sum_{i=1}^{n} (\theta_i' \odot p)(u_i \odot q))^{-1}.$$

On peut de plus supposer que $\theta_1 \neq 0$, $\theta_1' \neq 0$. Soit T_1 l'ensemble des points $t \in T$ tels que les fonctions θ_i, θ_i' soient définies en t et que $(\theta_1\theta_1')(t) \neq 0$; c'est une partie ouverte non vide de T, et, si $t \in T_1$, $j_t^*(\operatorname{div} w)$ est défini et égal au diviseur de la fonction $(\sum_{i=1}^{n} \theta_i(t) u_i)(\sum_{i=1}^{n} \theta_i'(t) u_i)^{-1}$. Il y a un diviseur $a \geqq 0$ sur C tel que $u_1, \cdots, u_n$ soient multiples de $-a$; si $c_1, \cdots, c_n$ sont des constantes non toutes nulles, $\operatorname{div} \sum_{i=1}^{n} c_i u_i$ se met sous la forme $a' - a$, où a' est un diviseur $\geqq 0$ de même degré que a, et est par suite de hauteur au plus égale au double du degré de a. Il en résulte que, si $t \in T_1$, $j_t^*(\operatorname{div} w)$ est de hauteur au plus égale au quadruple du degré de a.

Ceci étant, soit D le diviseur de définition de f. Il existe un recouvrement fini $(W_i)_{i \in I}$ de $T \times C$ par des ouverts $W_i \neq \emptyset$ tel que, pour chaque i, il existe une fonction numérique w_i sur $T \times C$ qui est fonction de définition de D en tout point de W_i. Pour chaque i, il existe une partie ouverte non vide T_i de T et un entier $h_i \geqq 0$ tels, pour tout $t \in T_i$, $j_t^*(\operatorname{div} w_i)$ soit défini et de hauteur $\leqq h_i$. L'intersection T' des T_i est une partie ouverte non vide de T. Soit t un point de cet ensemble; pour tout i, soit U_i l'ensemble des $x \in C$ tels que $(t, x) \in W_i$. Si x est un point de U_i, x intervient avec le même coefficient dans les cycles associés aux diviseurs $f(t)$ et $j_t^*(\operatorname{div} w_i)$, car $w_i \odot j_t$ est une fonction de définition de $f(t)$ en x. Il en résulte aussitôt que la hauteur de $f(t)$ est au plus égale à la somme des h_i pour tous les $i \in I$, ce qui établit la Proposition 3.

III. Criteres de rationalite (I). Nous allons indiquer une construction, essentiellement dûe à Cartier, qui permet d'associer à un diviseur sur un produit $T \times U$ et à un vecteur tangent à T un objet d'une nouvelle espèce, à savoir un diviseur additif sur U.

Soit U une variété; désignons par R le faisceau constant sur U dont les sections sur tout ouvert non vide U' sont les fonctions numériques sur U'.

Soit par ailleurs O le faisceau des anneaux locaux sur U; toute section du faisceau quotient R/O sur U s'appelle un *diviseur additif* sur U. Toute fonction numérique u sur U définit de manière évidente un diviseur additif, qu'on appelle le diviseur additif principal défini par u.

Ceci étant, soient T et U des variétés, D un diviseur, t un point de T tel que $j_t{}^*(D)$ soit défini (j_t étant l'application $x \rightarrow (t,x)$ de U dans $T \times U$) et L un vecteur tangent à T en t. Nous allons associer à D et à L un diviseur additif $< L, D >$ sur U. Soient p et q les projections de $T \times U$ sur T et sur U; pour chaque point $x \in U$, il y a un vecteur tangent bien determiné Λ_x à $T \times U$ en (t,x) dont les images par les applications dérivées de p et q en (t,x) sont L et 0 respectivement; nous dirons que $x \rightarrow \Lambda_x$ est le champ de vecteurs sur $\{t\} \times U$ défini par L. Soit w une fonction numérique sur $T \times U$ qui est définie en au moins un point de $\{t\} \times U$; pour tout $x \in U$ tel que w soit définie en (t,x), $< \Lambda_x, w >$ est un élément de K. Montrons que l'application $x \rightarrow < \Lambda_x, w >$ de l'ensemble ouvert des $x \in U$ tels que w soit définie en (t,x) se prolonge en une fonction numérique sur U, que nous désignerons par $< L, w >$. Soit x_0 un point de U tel que w soit définie en (t, x_0); w peut alors se mettre sous la forme $w'w''^{-1}$ où chacune des fonctions w', w'' est de la forme $\sum_{i=1}^{h}(\theta_i \odot p)(u_i \odot q)$, les θ_i étant des fonctions numériques sur T définies en t et les u_i des fonctions numériques sur U définies en x_0, et où, de plus, on a $w''(t,x_0) \neq 0$. Il existe un voisinage ouvert U_0 de x_0 dans U tel que chacune des fonctions u_i qui interviennent dans les expressions de w', w'' soit partout définie sur U_0 et que l'on ait $w''(t,x) \neq 0$ pour tout $x \in U_0$. Si x est un point de cet ensemble, on a

$$< \Lambda_x, w > = (w''(t,x))^{-2}(< \Lambda_x, w' > w''(t,x) - < \Lambda_x, w'' > w'(t,x)).$$

Par ailleurs, si $w_1 = \sum_{i=1}^{h} (\theta_i \odot p)(u_i \odot q)$, les θ_i étant définies et t et les u_i sur U_0, on a, pour $x \in U_0$, $< \Lambda_x, w_1 > = \sum_{i=1}^{h} < L, \theta_i > u_i(x)$. Il résulte de là que la restriction à U_0 de l'application $x \rightarrow < \Lambda_x, w >$ est une fonction numérique partout définie sur U_0. On en conclut que l'application $x \rightarrow < \Lambda_x, w >$ de l'ensemble U_1 des x tels que w soit définie en (t,x) est une fonction numérique partout définie sur U_1, ce qui démontre l'assertion faite plus haut.

On notera que, si h est un morphisme d'une variété T' dans la variété T et si L est l'image par la dérivée de h en un point $t' \in h^{-1}(t)$ d'un vecteur tangent L' à T' en t', on a $< L', w \odot h > = < L, w >$. En effet, considérant h comme définissant un morphisme de $T' \times U$ dans $T \times U$, le vecteur L'

définit un champ de vecteurs $(t', x) \to \Lambda_x'$ sur $T' \times U$ le long de $\{t'\} \times U$, et Λ_x n'est autre que l'image de Λ_x' par la dérivée de h en (t', x).

Ceci étant, soit (t, x) un point quelconque de $T \times U$, et soit w une fonction de définition de D en (t, x) ; w est alors définie en au moins un point de $\{t\} \times U$. La classe de la fonction $(w \odot j_t)^{-1} < L, w >$ modulo l'anneau local $\mathfrak{o}(x)$ de x ne dépend pas du choix de w. En effet, si w_1 est une autre fonction de définition de D en (t, x), $w^{-1}w_1 = z$ est une fonction définie en (t, x) et y prenant une valeur $\neq 0$, de sorte que $(z \odot j_t)^{-1} < L, z >$ est dans $\mathfrak{o}(x)$; or on a

$$(w_1 \odot j_t)^{-1} < L, w_1 > = (w \odot j_t)^{-1} < L, w > + (z \odot j_t)^{-1} < L, z >,$$

ce qui établit notre assertion. Désignons par δ_x la classe de $(w \odot j_t)^{-1} < L, w >$ modulo $\mathfrak{o}(x)$; il est clair que, pour tous les points x' d'un voisinage convenable de x, $\delta_{x'}$ est aussi la classe de $(w \odot j_t)^{-1} < L, w >$ modulo $\mathfrak{o}(x')$. Ceci montre que l'application $x \to \delta_x$ est un diviseur additif; nous le noterons $< L, D >$. Il est clair que l'on a

$$< L, D + D' > = < L, D > + < L, D' >$$

si D et D' sont des diviseurs sur $T \times U$ tels que $j_t^*(D)$ et $j_t^*(D')$ soient définis. Si w est une fonction numériqe sur $T \times U$ définie en au moins un point de $\{t\} \times U$, $< L, \operatorname{div} w >$ est le diviseur additif associé à la fonction numérique $(w \odot j_t)^{-1} < L, w >$ sur U.

LEMME 1. *Soient T, T', U variétés, h un morphisme de T' dans T, t' un point de T', t le point $h(t')$, L' un vecteur tangent à T' en t', L l'image de L' par l'application dérivée de h, D un diviseur sur $T \times U$ tel que $h^*(D)$ et $j_t^*(D)$ soient définis (j_t étant l'application $x \to (t, x)$ de U dans $T \times U$). On a alors $< L', h^*(D) > = < L, D >$.*

Si $x \in U$ et si w est fonction de définition de D en (t, x), $w \odot h$ est fonction de définition de $h^*(D)$ en (t', x) ; le Lemme 1 résulte alors de ce qui a été dit plus haut.

Si Δ est un diviseur additif sur U, la valeur de Δ en un point x est une classe modulo l'anneau local de x; tout représentant de cette classe s'appelle une fonction de définition de Δ en x.

LEMME 2. *Soient T et U des variétés, t un point de T, L un vecteur tangent à T en t, D un diviseur $\geqq 0$ sur $T \times U$ tel que $j_t^*(D)$ soit défini. Soient u une fonction de définition de $j_t^*(D)$ en un point $x \in U$ et s une fonction de définition du diviseur additif $< L, D >$ en x; la fonction su est alors définie en x.*

Soit w une fonction de définition de D en (t, x) ; on peut supposer que $u = w \odot j_t$ et que $s = (w \odot j_t)^{-1} < L, w >$; le lemme résulte alors de ce que, w étant définie en (t, x), $< L, w >$ est définie en x.

Si f est une famille algébrique de diviseurs d'une variété U paramétrée par une variété T, si $t \in T$ et si L est un vecteur tangent à T en t, nous poserons $< L, f > = < L, D >$, D étant le diviseur de définition de f. Si on a $< L, f > \neq 0$ pour tout vecteur tangent $L \neq 0$ à T en t, on dit que f est *infinitésimalement injective en t*. Si f est infinitésimalement injective en tout point de T, on dit que cette famille est *infinitésimalement injective.*

Proposition 2. *Soit f une famille injective de diviseurs d'une variété U paramétrée par une variété complète T. Soit f' une famille de diviseurs de U paramétrée par une variété normale T'; supposons que, pour tout $t' \in T'$ il existe un point $t \in T$ tel que $f(t) = f'(t')$ et qu'il existe un point $(t_0, t'_0) \in T \times T'$ tel que $f(t_0) = f'(t'_0)$ et que f soit infinitésimalement injective en t_0. Il existe alors un morphisme h de T' dans T tel que $f' = f \circ h$.*

L'ensemble E des points $(t, t') \in T \times T'$ tels que $f(t) = f'(t')$ est fermé (Corollaire 4 au Théorème 1, § I), et il résulte des hypothèses que la projection $T \times T' \to T'$ induit une bijection p de E sur T'. Montrons que E est irréductible. Il existe au moins une composante irréductible E_1 de E telle que $p(E_1)$ soit dense dans T'. Or, T étant complète, la projection $T \times T' \to T'$ est une application propre; comme E_1 est fermé, on a $p(E_1) = T'$, d'où $E_1 = E$, puisque p est bijectif, ce que établit notre assertion. L'application p est donc un morphisme bijectif propre de E sur T'; si nous montrons qu'il est birationnel, il résultera du théorème principal de Zariski et du fait que T' est normale que p est un isomorphisme de E sur T'. En composant l'isomorphisme p^{-1} de T' sur E avec la restriction à E de la projection $T \times T' \to T$, on obtiendra un morphisme h possédant la propriété requise.

Or, le morphisme p, qui est bijectif, est radiciel; pour montrer qu'il est birationnel, il suffira de montrer qu'il est séparable, donc qu'il y a un point de E en lequel p n'est pas ramifié (i. e. tel que l'application dérivée de p en ce point soit injective). Nous allons voir que le point (t_0, t'_0) possède cette propriété. Soit L'' un vecteur tangent $\neq 0$ à E en (t_0, t'_0) ; soient L et L' les images de L'' par les applications dérivées des restrictions q et p à E des projections de $T \times T'$ sur T et sur T'; comme L'' s'identife à un vecteur tangent à $T \times T'$, L et L' ne sont pas tous deux nuls; nous voulons montrer que $L \neq 0$. Il est clair que l'on a $f \circ q = f' \circ p$; soit f'' leur valeur comme. Il résulte du Lemme 1 que $< L'', f'' > = < L, f > = < L', f' >$; si on avait $L' = 0$, on aurait $L \neq 0$, d'où $< L, f > \neq 0$ puisque f est infinitésimalement injective en t_0, d'où contradiction. La Proposition 2 est donc établie.

Nous allons maintenant donner des exemples de familles infinitésimalement injectives. Soit U une variété *semi-complète,* et soit V un sous-espace vectoriel de dimension finie >0 de l'espace des fonctions numériques sur U; V définit alors un système linéaire f de diviseurs de U paramétré par $\mathfrak{P}(V)$. Nous allons voir que f est infinitésimalement injectif. Désignons par V_0 l'ensemble des éléments $\neq 0$ de V et par φ l'application canonique de V_0 sur $\mathfrak{P}(V)$; si $u_0 \in V_0$, l'application dérivée de φ en u_0 est une surjection de l'espace tangent à V_0 en u_0 sur l'espace tangent à $\mathfrak{P}(V)$ en $\varphi(u_0)$. Tenant compte du Lemme 1, on voit qu'il suffira de montrer que tout vecteur tangent L à V_0 en u_0 tel que $< L, f\circ\varphi > = 0$ appartient au noyau de la dérivée de φ. Or le diviseur de définition de $f\circ\varphi$ est $\operatorname{div} w$, où w est la fonction numérique sur $V_0 \times U$ telle que $w(u,x) = u(x)$ si $u \in V_0$ et si u est définie en x. Soit $(u_0, u_1, \cdots, u_m)$ une base de V contenant u_0; soient $\lambda_0, \cdots, \lambda_m$ les coordonnées relativement à cette base; si donc x est un point en lequel $u_0, \cdots, u_m$ sont définies, on a $w(u,x) = \sum_{i=0}^{m} \lambda_i(u) u_i(x)$, d'où il résulte tout de suite que $< L, w >$ est la fonction $\sum_{i=0}^{m} < L, \lambda_i > u_i$, et que $< L, \operatorname{div} w >$ est le diviseur additif représenté par la fonction $\sum_{i=0}^{m} < L, \lambda_i > u_0^{-1} u_i$. Si ce diviseur additif est nul, la fonction $\sum_{i=0}^{m} < L, \lambda_i > u_0^{-1} u_i$ est partout définie, donc constante puisque U est semi-complète; comme $u_0, \cdots, u_m$ sont linéairement indépendantes, ceci n'est possible que si $< L, \lambda_i > = 0$ pour $0 \leqq i \leqq m$; or ceci est précisément la condition pour que l'image de L par la dérivée de φ soit nulle.

Soit maintenant C une courbe normale, et soit r un entier >0; soit d_r la famille canonique de diviseurs de C paramétrée par la puissance symétrique r-ième S^r de C, et soit s_r l'application canonique de C^r sur S^r. Nous allons montrer que d_r est infinitésimalement injective en tout point de S^r de la forme $s_r(a_1, \cdots, a_r)$, $a_1, \cdots, a_r$ étant des points *tous distincts* de C [en fait, on peut montrer que d_r est infinitésimalement injective en tout point de S^r, mais le raisonnement est plus compliqué]. Le morphisme s_r n'est pas ramifié au point $(a_1, \cdots, a_r)$, qui est simple sur C^r; sa dérivée en ce point est donc un isomorphisme de l'espace tangent à C^r en $(a_1, \cdots, a_r)$ sur l'espace tangent à S^r au point $s_r(a_1, \cdots, a_r)$. Il nous suffira donc de montrer que l'on a $< L, d_r \circ s_r > \neq 0$ si L est un vecteur tangent $\neq 0$ à C^r en $(a_1, \cdots, a_r)$. Soient $q_1, \cdots, q_r, q_{r+1}$ les projections de $C^{r+1} = C^r \times C$ sur les divers facteurs. Soient $L_1, \cdots, L_r$ les images de L par les dérivées des projections de C^r sur ses divers facteurs; L_i est donc un vecteur tangent à C en a_i, et il existe au moins un i, soit k, tel que $L_i \neq 0$. Soit u une variable uniformisante en a_k sur C. L'application $d_r \circ s_r$ est l'application $(x_1, \cdots, x_r) \to \sum_{i=1}^{r} x_i$; tenant

compte de ce que $a_1, \cdots, a_r$ sont mutuellement distincts, on voit tout de suite que la fonction $w = u \odot q_k - u \odot q_{r+1}$ est fonction de définition en $((a_1, \cdots, a_r), a_k)$ du diviseur de définition de $d_r \circ s_r$. Il est clair que $< L, w >$ est la fonction constante $< L_k, u >$ sur C, de sorte que la fonction $(u(a_k) - u)^{-1} < L_k, u >$ est fonction de définition de $< L, d_r \circ s_r >$ en a_k. Or, on a $< L_k, u > \neq 0$ puisque u est variable uniformisante en a_k et $L_k \neq 0$; comme a_k est un zéro de $u(a_k) - u$, on voit que $< L, d_r \circ s_r > \neq 0$, ce qui établir notre assertion.

IV. Familles algebriques de classes de diviseurs. Pour toute variété X, nous désignerons par $\mathfrak{D}(X)$ le groupe des diviseurs de X, par $\mathfrak{P}(X)$ le groupe des diviseurs principaux de X et par $\mathfrak{G}(X) = \mathfrak{D}(X)/\mathfrak{P}(X)$ le groupe des classes de diviseurs de X.

PROPOSITION 1. *Soit $\mathfrak{k}$ une classe de diviseurs sur une variété X; si x_0 est un point de X, il existe dans $\mathfrak{k}$ un diviseur dont le support ne contient pas x_0.*

Soient en effet d un diviseur quelconque de la classe $\mathfrak{k}$ et u une fonction de définition de d en x_0; $d' = d - \operatorname{div} u$ possède alors la propriété requise.

Soit f un morphisme d'une variété V dans une variété U. Si $\mathfrak{k} \in \mathfrak{G}(U)$, il existe toujours un diviseur $d \in \mathfrak{k}$ tel que $f^*(d)$ soit défini; il suffit en effet de choisir un point $y_0 \in V$ et un diviseur $d \in \mathfrak{k}$ tel que $f(y_0) \notin \operatorname{Supp} d$. De plus, les diviseurs $f^*(d)$, pour tous les diviseurs $d \in \mathfrak{k}$ tels que $f^*(d)$ soit défini, appartiennent tous à une même classe. Pour le voir, il suffit de montrer que, si d est un diviseur principal tel que $f^*(d)$ soit défini, $f^*(d)$ est principal. Or, si $d = \operatorname{div} u$, il y a au moins un point $y \in V$ tel que u et u^{-1} soient définies en $f(y)$, de sorte que $u \odot f$ est définie et $\neq 0$; il résulte alors tout de suite des définitions que $f^*(d) = \operatorname{div} u \odot f$. Nous désignerons par $f^*(\mathfrak{k})$ la classe de diviseurs de V qui contient les $f^*(d)$ pour les $d \in \mathfrak{k}$ tels que $f^*(d)$ soit défini. Il est clair que l'application f^* ainsi définie est un homomorphisme de $\mathfrak{G}(U)$ dans $\mathfrak{G}(V)$. Soit maintenant g un morphisme d'une variété W dans la variété V; on a alors $g^*(f^*(\mathfrak{k})) = (f \circ g)^*(\mathfrak{k})$ pour tout $\mathfrak{k} \in \mathfrak{G}(U)$. En effet, soit d un représentant de $\mathfrak{k}$ tel que $(f \circ g)^*(d)$ soit défini; alors $f^*(d)$ et $g^*(f^*(d))$ sont définis, et $g^*(f^*(d)) = (f \circ g)^*(d)$, ce qui démontre notre assertion. Il résulte de là que l'application $U \to \mathfrak{G}(U)$ définit un foncteur contravariant sur la catégories des variétés à valeurs dans celle des groupes abéliens.

Soient en particulier T et U des variétés, et $\mathfrak{k}$ un élément de $\mathfrak{G}(T \times U)$. Pour tout $t \in T$, soit j_t l'application $x \to (t, x)$ de U dans $T \times U$; l'application

$f: t \to j_t^*(\mathfrak{k})$ est alors une application de T dans $\mathfrak{G}(U)$. On appelle *famille algébrique des classes de diviseurs de U paramétrée par T* (ou application algébrique de T dans $\mathfrak{G}(U)$) toute application de T dans $\mathfrak{G}(U)$ qui peut se définir de la manière qu'on vient d'indiquer à partir d'une classe de diviseurs sur $T \times U$.

Pour tout diviseur d sur une variété, nous désignerons par Cl. d la classe de diviseurs de d. Si $\bar{f}$ est une famille algébrique de diviseurs d'une variété U paramétrée par une variété T, l'application $t \to \mathrm{Cl}.\bar{f}(t)$ est évidemment une famille algébrique de classes de deviseurs.

PROPOSITION 2. *Soit f une famille algébrique de classes de diviseurs d'une variété U paramétrée par une variété T. Si $t_0 \in T$, il existe un voisinage ouvert T_0 de t_0 et une famille algébrique $\bar{f}$ de diviseurs de U paramétrée par T_0 tels que l'on ait $f(t) = \mathrm{Cl}.\bar{f}(t)$ pour tout $t \in T_0$.*

Soit x_0 un point de U. La famille f est définie par une classe $\mathfrak{k} \in \mathfrak{G}(T \times U)$, et $\mathfrak{k}$ contient un diviseur D dont le support ne passe pas par (t_0, x_0). Il existe un voisinage ouvert T_0 de t_0 tel que $T_0 \times \{x_0\}$ ne rencontre pas Supp D; soit D_0 le diviseur induit par D sur $T_0 \times U$. Il est clair que, si $t \in T_0$, $\{t\} \times U$ n'est pas contenu dans Supp D_0; D_0 définit donc une famille algébrique $\bar{f}$ de diviseurs de U paramétrée par T_0, qui possède évidemment la propriété requise.

Observons maintenant que, si une classe $\mathfrak{k} \in \mathfrak{G}(T \times U)$ définit une famille f de classes de diviseurs de U, il n'est pas nécessaire que l'on ait $\mathfrak{k} = 0$ pour que l'on ait $f = 0$. Soit en effet p la projection de $T \times U$ sur son premier facteur; supposons que $\mathfrak{k} \in p^*(\mathfrak{G}(T))$; on va montrer que l'on a alors $f = 0$. Soit $\mathfrak{k} = p^*(\mathfrak{k}_1)$, $\mathfrak{k}_1 \in \mathfrak{G}(T)$. Soit $t \in T$; il y a dans $\mathfrak{k}_1$ un diviseur d_1 dont le support ne contient pas t; $d_1 \times U$ est alors un représentant de $\mathfrak{k}$; j_t^* étant défini comme plus haut, $j_t^*(d_1 \times U)$ est défini et nul puisque $t \notin \mathrm{Supp}\, d_1$; il en résulte que $f(t) = 0$ quel que soit t. Ceci conduit à introduire le groupe

$$\mathfrak{M}(T; U) = \mathfrak{G}(T \times U)/p^*(\mathfrak{G}(T));$$

si m est un élément de ce groupe, toutes les classes $\mathfrak{k} \in m$ définissent la même famille f de classes de diviseurs de U; on dit que f est la famille définie par m. Nous verrons tout à l'heure que, sous certaines hypothèses, la condition $f = 0$ entraîne $m = 0$.

Si h est un morphisme d'une variété T' dans une variété T, et si on désigne par p' la projection de $T' \times U$ sur $T \times U$, il est clair que l'homomorphisme h^* de $\mathfrak{G}(T \times U)$ dans $\mathfrak{G}(T' \times U)$ applique $p^*(\mathfrak{G}(T))$ dans $p'^*(\mathfrak{G}(T'))$, et définit par suite un homomorphisme de $\mathfrak{M}(T, U)$ dans $\mathfrak{M}(T', U)$.

Désignons encore par q la projection de $T \times U$ sur son deuxième facteur U. Il nous sera commode d'introduire aussi le groupe

$$\mathfrak{N}(T,U) = \mathfrak{G}(T \times U)/(p^*(\mathfrak{G}(T)) + q^*(\mathfrak{G}(U)));$$

ici encore, à tout morphisme d'une variété T' dans la variété T est attaché un homomorphisme de $\mathfrak{N}(T,U)$ dans $\mathfrak{N}(T',U)$, de sorte que, pour U fixe, les groupes $\mathfrak{N}(T,U)$ pour toutes les variétés T définissent un facteur contravariant sur la catégorie des variétés. On notera que l'isomorphisme canonique de $U \times T$ sur $T \times U$ définit un *isomorphisme canonique*

$$\mathfrak{N}(T,U) \cong \mathfrak{N}(U,T).$$

On notera que, si une classe $\mathfrak{f}$ de diviseurs de $T \times U$ appartient au groupe $q^*(\mathfrak{G}(U))$, l'application algébrique f de T dans $\mathfrak{G}(U)$ qu'elle définit est constante; en effet, $\mathfrak{f}$ contient alors un diviseur de la forme $T \times d$, où d est un diviseur de U; si t est un point quelconque de T, $f(t)$ est la classe du diviseur d.

THÉORÈME 2. *Soit f une famille algébrique de classes de diviseurs d'une variété semi-complète U paramétrée par une variété T. L'ensemble des points $t \in T$ tels que $f(t) = 0$ est alors fermé.*

Nous établirons d'abord le lemme suivant:

LEMME 1. *Soient T et U des variétés, $\mathfrak{f}$ une classe de diviseurs de $T \times U$, t_0 un point de T. Il existe alors un diviseur D de la classe $\mathfrak{f}$, un voisinage ouvert T_0 de t_0 dans T et un faisceau A d'idéaux fractionnaires sur U qui possèdent les propriétés suivantes: si $t \in T_0$, $j_t^*(D) = \delta(t)$ est défini (j_t étant l'application $x \to (t,x)$ de U dans $T \times U$), et le faisceau d'idéaux $A^{\delta(t)}$ est un sous-faisceau de A.*

Soit d'abord D_1 un représentant quelconque de la classe $\mathfrak{f}$ tel que $j_{t_0}^*(D_1)$ soit défini. Il existe un voisinage affine T_1 de t_0 tel que $j_t^*(D_1)$ soit défini pour tout $t \in T_1$. Soit U_0 un morceau affine quelconque de U. Les sections sur $T_1 \times U_0$ du faisceau d'idéaux fractionnaires sur $T \times U$ associé à D_1 forment un idéal fractionnaire $\mathfrak{D}$ pour l'algèbre affine $\mathfrak{O}$ de $T_1 \times U_0$. Soit $\mathfrak{T}$ l'idéal de $\mathfrak{O}$ composé des éléments $w \in \mathfrak{O}$ tels que $w\mathfrak{D} \subset \mathfrak{O}$. Si $(t,x) \in T_1 \times U_0$, l'idéal engendré par $\mathfrak{T}$ dans l'anneau local $\mathfrak{o}(t,x)$ de (t,x) n'est autre que l'ensemble des éléments $w \in \mathfrak{o}(t,x)$ tels que $w(\mathfrak{D}\mathfrak{o}(t,x)) \subset \mathfrak{o}(t,x)$. Puisque $j_{t_0}^*(D_1)$ est défini, il y a un point $x_0 \in U_0$ tel que $\mathfrak{D}\mathfrak{o}(t_0,x_0) = \mathfrak{o}(t_0,x_0)$; il y a donc un élément $w \in \mathfrak{T}$ tel que $w(t_0,x_0) \neq 0$. Posons $D = D_1 + \operatorname{div} w$; D est encore un représentant de la classe $\mathfrak{f}$; il est clair que D est positif en

tout point de $T_1 \times U_0$; de plus, $j_{t_0}(D)$ est défini. Il y a donc un voisinage ouvert T_0 de t_0 tel que $j_t{}^*(D)$ soit défini toutes les fois que $t \in T_0$. Soit D' le diviseur induit par D sur $T_0 \times U$; si nous posons $E = T_0 \times (U - U_0)$, D' est positif en tout point de $T_0 \times (U - E)$. Si donc B est le faisceau d'idéaux qui définit l'ensemble fermé E, il y a un entier $k \geqq 0$ tel que $B^k A^{D'}$ soit un faisceau d'idéaux entiers, $A^{D'}$ désignant le faisceou d'idéaux sur $T_0 \times U$ associé à D' (cf. § I). Désignons par C le faisceou d'idéaux sur U qui définit l'ensemble $U - U_0$, et par A le transporteur de C^k dans le faisceau $\mathfrak{O}$ des anneaux locaux de U. Nous allons montrer que, si $t \in T_0$, $\delta(t) = j_t{}^*(D)$, le faisceau d'idéaux associé à $\delta(t)$ est un sous-faisceau de A. Soient x un point de U, w une fonction de définition de D en (t,x) et $u = w \odot j_t$; u est donc fonction de définition de $\delta(t)$ en x. Pour montrer que l'idéal ponctuel en x du faisceau associé à $\delta(t)$ est contenu dans A_x, il suffit de montrer que, si $v_1, \cdots, v_k$ sont des fonctions de l'idéal ponctuel C_x de C en x, $v_1 \cdots v_k u$ appartient à l'anneau local de x. Soit q la projection $T \times U \to U$; chacune des fonctions v_i est nulle sur l'intersection avec $U - U_0$ d'un voisinage convenable de x, de sorte que les fonctions $v_i \odot q$ sont nulles sur l'intersection de E avec un voisinage convenable de (t,x), ce qui montre qu'elles appartiennent à l'idéal ponctuel de B en (t,x); la fonction $(v_1 \cdots v_k \odot q)w$ est donc définie en (t,x). Or on a $v_i = (v_i \odot q) \odot j_t$, d'où

$$v_1 \cdots v_k u = ((v_1 \cdots v_k \odot q)w) \odot j_t,$$

ce qui établit notre assertion. Le Lemme 1 est donc établi.

Ceci établi, nous pouvons démontrer le Théorème 2. L'application f est définie au moyen d'une classe $\mathfrak{k}$ de diviseurs de $T \times U$; soit t_0 un point de T adhérent à l'ensemble H des points t tels que $f(t) = 0$; nous choisirons T_0, D, A comme dans le Lemme 2. Puisque U est semi-complète, les sections de A forment un espace vectoriel V de dimension finie. Si $t \in H \cap T_0$, $\delta(t)$, qui est un représentant de la classe $f(t)$, est un diviseur principal; c'est donc le diviseur d'une fonction qui appartient à V. Ceci montre que $V \neq \{0\}$; soit $\zeta \to \operatorname{div} \zeta$ le système linéaire paramétrée par l'espace projectif $\mathfrak{P}(V)$ associé à V. Alors $H \cap T_0$ est l'image par la projection de $T_0 \times \mathfrak{P}(V)$ sur T_0 de l'ensemble L des couples (t,ζ) tels que $f(t) = \operatorname{div} \zeta$ (car, pour tout $\zeta \in \mathfrak{P}(V)$, $\operatorname{div} \zeta$ est principal). Or, L est fermé (Corollaire 4 au Théorème 1, § II); $H \cap T_0$ est donc relativement fermé dans T_0, d'où $t_0 \in H$, ce qui démontre le Théorème 2.

THÉORÈME 3. *Soient T une variété normale, U une variété semi-complète et m un élément de $\mathfrak{M}(T,U)$; la famille algébrique de classes de diviseurs définie par m ne peut être nulle que si $m = 0$.*

Reprenons les notations de la démonstration du Théorème 2, t_0 étant ici un point quelconque de T. On a vu au § III que le système linéaire $\zeta \rightarrow \operatorname{div} \zeta$ est une famille injective et infinitésimalement injective. Il existe donc un morphisme g de T_0 dans $\mathfrak{P}(V)$ tel que l'on ait $\delta(t) = \operatorname{div} g(t)$ pour $t \in T_0$. Soit φ l'application canonique de l'ensemble V_0 des éléments $\neq 0$ de V sur $\mathfrak{P}(V)$; il existe un morphisme ψ d'un voisinage P de $g(t_0)$ dans V_0 tel que $\varphi \circ \psi$ soit l'application identique de P. Soit T_0' un voisinage ouvert de t_0 tel que $g(T_0') \subset P$, et soit g' l'application $t \rightarrow \psi(g(t))$ de T_0' dans V_0 ; on a donc $\delta(t) = \operatorname{div} g'(t)$ si $t \in T_0'$ (rappelons que $g'(t)$, en tant qu'élément de V_0, est une fonction numérique sur U). Si U_1 est l'ensemble des points de U en lesquels sont définies toutes les fonctions de V, il est clair que $(t, x) \rightarrow (g'(t))(x)$ est une fonction numérique sur $T_0' \times U_1$ qui se prolonge en une fonction numérique w sur $T \times U$; si $t \in T_0'$ on a

$$j_t^*(\operatorname{div} w) = \operatorname{div} g'(t) = \delta(t) = j_t^*(D),$$

j_t désignant l'application $x \rightarrow (t, x)$ de U dans $T \times U$. On en conclut que le diviseur induit par $D - \operatorname{div} w$ sur $T_0' \times U$ définit la famille nulle de diviseurs de U paramétrée par T_0', donc est nul. Or $D' = D - \operatorname{div} w$ est un représentant de la classe $\mathfrak{k}$ de diviseurs de $T \times U$ qui contient D. Il résulte de là que $\operatorname{Supp} D' \subset (T - T_0') \times U$. Le Théorème 3 résultera donc du

Lemme 2. *Soient T une variété normale, U une variété et D un diviseur sur $T \times U$. S'il existe une partie fermée $E \neq T$ de T telle que* $\operatorname{Supp} D \subset E \times U$, *$D$ est de la forme $d \times U$, d étant un diviseur de T.*

Nous considérerons d'abord le cas où U est aussi supposée normale. Pour tout point $x \in U$, nous désignerons par k_x l'application $t \rightarrow (t, x)$ de T dans $T \times U$. Nous allons montrer que, si $D \neq 0$, on a $k_x^*(D) \neq 0$ pour tout $x \in U$. Puisque $T \times U$ est normale, toute composante irréductible Σ de $\operatorname{Supp} D$ est une hypersurface de $T \times U$; comme $\Sigma \subset E \times U$, il est clair que Σ est de la forme $S \times U$, S étant une hypersurface de T. Soient $S_1, \cdots, S_h$ les hypersurfaces de T telles que $S_i \times U \subset \operatorname{Supp} D$ (avec $S_i \neq S_j$ si $i \neq j$) ; soit t un point de S_1 qui n'appartient à aucun S_i d'indice $i > 1$. Soit w une fonction de définition de D en (t, x) ; $S_1 \times U$ est donc la seule hypersurface de $T \times U$ passant part (t, x) qui soit ou bien variété de pôles ou bien variété de zéros de w. La variété $T \times U$ étant normale, il en résulte que, si (t, x) est pôle de w, w^{-1} est définie en (t, x) et y prend la valeur 0, tandisque, si (t, x) est un zéro de w, w est définie en ce point et y prend la valeur 0. On en conclut que, dans le premier cas, t est un pôle de $w \odot k_x$, tandisque, dans le second cas, c'en est un zéro; dans les deux cas, t appartient à $k_x^*(D)$. Ceci

étant, soit x_0 un point quelconque de U ; posons $d = k_{x_0}{}^*(D)$, $D' = D - d \times U$; il est clair que $\operatorname{Supp} D' \subset (E \cup \operatorname{Supp} d) \times U$; de plus, on a $k_{x_0}{}^*(D') = 0$; en vertu de ce qu'on vient d'établir, il en résule que $D' = 0$. Pour passer au cas général, désignons par U_0 l'ensemble des points de U en lesquels U est normale ; c'est une sous-variété ouverte et normale de U. Il en résulte que le diviseur D_0 induit par D sur $T \times U_0$ se met sous la forme $d \times U_0$, d étant un diviseur sur T ; on a donc $k_x{}^*(D - d \times U) = 0$ pour tout $x \in U_0$. Or, $x \to k_x{}^*(D - d \times U)$ est une famille algébrique de diviseurs de T paramétrée par U ; comme sa restriction à la partie dense U_0 de U est nulle, son diviseur de définition $D - d \times U$ est nul, ce qui démontre le Lemme 2.

Soient T une variété normale et U une variété; désignons par p la projection $T \times U \to T$. Si T' et T'' sont des parties ouvertes non vides de T avec $T' \subset T''$, l'injection canonique $T' \to T''$ définit des homomorphismes $\mathfrak{D}(T'' \times U) \to \mathfrak{D}(T' \times U)$, $\mathfrak{P}(T'' \times U) \to \mathfrak{P}(T' \times U)$, $\mathfrak{D}(T'') \to \mathfrak{D}(T')$, $p^*(\mathfrak{D}(T'')) \to p^*(\mathfrak{D}(T'))$. Les applications

$$T' \to \mathfrak{D}(T' \times U), \; T' \to \mathfrak{P}(T' \times U), \; T' \to p^*(\mathfrak{D}(T'))$$

sont donc munies de structures de pré-faisceaux de groupes commutatifs sur T. Il est évident que l'application $T' \to \mathfrak{D}(T' \times U)$ est un faisceau, que nous désignerons par $\mathfrak{D}_U$. Montrons que l'application $T' \to \mathfrak{P}(T' \times U) + p^*(\mathfrak{D}(T'))$ est un sous-faisceau de $\mathfrak{D}_U$. Sachant déjà que $\mathfrak{D}_U$ est un faisceau, il suffit de vérifier ce qui suit: soit D' un diviseur sur $T' \times U$; supposons qu'il existe un recouvrement $(T'_i)_{i \in I}$ de T' par des ouverts non vides T'_i tels que, pour tout i, le diviseur D'_i induit par D' sur $T'_i \times U$ soit de la forme $\operatorname{div} w_i + d' \times U$, où w_i est une fonction numérique sur $T'_i \times U$ et d'_i un diviseur sur T'_i ; alors D est de la forme $\operatorname{div} w' + d'$, w' étant une fonction numérique sur $T' \times U$ et d' un diviseur sur T'. Chacune des fonctions w_i se prolonge en une fonction numérique sur $T' \times U$, que nous désigerons encore par w_i. Soit i_0 un élément de I ; remplaçant D' par $D' - \operatorname{div} w_{i_0}$, nous nous ramenons au cas où $\operatorname{div} w_{i_0} = 0$. Pour chaque i, le diviseur induit par D'_i sur $(T'_i \cap T'_{i_0}) \times U$ est alors identique à celui qui est induit par $d'_{i_0} \times U$ sur ce même ensemble ; il en résulte que, si E_i est la réunion de $\operatorname{Supp} d'_{i_0}$ et de $T' - (T'_i \cap T'_{i_0})$, on a $\operatorname{Supp} D'_i \subset E_i \times U$, d'où il résulte, en vertu du Lemme 2, que D'_i est de la forme $d''_i \times U$, d''_i étant un diviseur sur T'_i. Il est clair que, pour toute sous-variété ouverte T'' de T, l'application $d'' \to d'' \times U$ de $\mathfrak{D}(T'')$ dans $\mathfrak{D}(T'' \times U)$ est injective ; on en conclut que, si $i, j \in I$, d''_i et d''_j induisent le même diviseur sur $T'_i \cap T'_j$. Il existe donc un diviseur d' sur T' tel que, pour tout i, d'_i soit le diviseur induit par d' sur T'_i. Le diviseur $D' - d' \times U$ induit alors 0 sur chacun des $T'_i \times U$ et est par suite nul, ce qui démontre notre assertion.

Si T' est une partie ouverte non vide de T, on a

$$\mathfrak{M}(T',U) = \mathfrak{D}(T' \times U)/(\mathfrak{P}(T' \times U) + p^*(\mathfrak{D}(T'))).$$

L'application $T' \to \mathfrak{M}(T',U)$ est évidemment munie d'une structure de pré-faisceau; ce pré-faisceau est le quotient *dans la catégorie des pré-faisceaux* du faisceau $\mathfrak{D}_U$ par le faisceau $\mathfrak{Q}_U$: $T' \to \mathfrak{P}(T' \times U) + p^*(\mathfrak{D}(T'))$. Il n'en résulte pas que $T' \to \mathfrak{M}(T',U)$ soit un faisceau. Cependant, observons que, si T est une variété non singulière, $\mathfrak{Q}_U$ est un faisceau *flasque*. En effet, soient alors T' une partie ouverte non vide de T, w' une fonction numérique sur $T' \times U$ et d' un diviseur sur T'; w' se prolonge alors en une fonction numérique w sur $T \times U$, et div w' est le diviseur induit par div w sur $T' \times U$; par ailleurs, T étant non singulière, d' est induit sur T' par un diviseur d sur T (Corollaire à la Proposition 1, § I); il en résulte que div $w' + d' \times U$ est le diviseur induit par div $w + d \times U$ sur $T' \times U$, ce qui montre bien que $\mathfrak{Q}_U$ est flasque. Or le pré-faisceau quotient d'un faisceau par un sous-faisceau flasque est un faisceau ([4], Théorème 3.1.2, p. 148); on voit donc que, si T est non singulière, $T' \to \mathfrak{M}(T',U)$ est un faisceau de groupes commutatifs. Si on suppose U semi-complète, il y a correspondance biunivoque entre éléments de $\mathfrak{M}(T,U)$ et familles algébriques de classes de diviseurs paramétrées par T; on obtient donc le résultat suivant:

PROPOSITION 3. *Soient T une variété non singulière et U une variété semi-complète; soit f une application de T dans $\mathfrak{G}(U)$. Supposons que tout point $t \in T$ admette un voisinage ouvert T' tel que la restriction de f à T' soit une famille algébrique paramétrée par T'; f est alors une famille algébrique paramétrée par T.*[4]

COROLLAIRE. *Les notations étant celles de la Proposition 3, supposons que tout point de T admette un voisinage ouvert T' qui possède la propriété suivante: il existe une famille algébrique f' de diviseurs de U paramétrée par T' telle que $f(t) = \mathrm{Cl}. f'(t)$ pour tout $t \in T'$. Alors f est une famille algébrique paramétrée par T.*

Soient T et U des variétés, p et q les projections de $T \times U$ sur T et sur U. Alors $q^*(\mathfrak{G}(U))$ est un sous-groupe de $\mathfrak{G}(T \times U)$; désignons par $\mathfrak{M}_0(T,U)$ son image canonique dans $\mathfrak{M}(T,U) = \mathfrak{G}(T \times U)/p^*(\mathfrak{G}(T))$; on a donc $\mathfrak{N}(T,U) = \mathfrak{M}(T,U)/\mathfrak{M}_0(T,U)$; de plus, on a une application surjective $\mathfrak{G}(U) \to \mathfrak{M}_0(T,U)$; montrons que cette application est même un isomorphisme. Il suffit de montrer que, si $\mathfrak{c}$ est une classe de diviseurs de U telle

[4] Serre m'a communiqué une démonstration du fait que l'hypothèse "T non singulière" est superflue dans l'énoncé de la Proposition 3.

que $q^*(\mathfrak{c})$ appartienne à $p^*(\mathfrak{G}(T))$, on a $\mathfrak{c}=0$. Par hypothèse, la classe $q^*(\mathfrak{c})$, qui contient un diviseur de la forme $T \times e$, e étant un diviseur sur U, contient aussi un diviseur de la forme $d \times U$, d étant un diviseur sur T; on a donc $T \times e = d \times U + P$, où P est un diviseur principal. Soit t un point de T n'appartenant pas à $\operatorname{Supp} d$, et soit j l'application $x \to (t, x)$ de U dans $T \times U$; alors $j^*(T \times e)$ est défini et égal à e, et $j^*(d \times U)$ est défini et nul; $j^*(P)$ est donc défini et égal à e. Or, comme P est principal, il en est de même de $j^*(P)$, donc de e, d'où $\mathfrak{c}=0$.

Si nous supposons que T est non singulière, l'application $T' \to \mathfrak{M}(T', U)$ (T' ouvert non vide dans T) est, comme on l'a vu, un faisceau. Il résulte immédiatement de ce qu'on vient de dire que $T' \to \mathfrak{M}_0(T', U)$ est un sous-faisceau constant du faisceau $T' \to \mathfrak{M}(T', U)$, isomorphe au faisceau constant de valeur $\mathfrak{G}(U)$. Un faisceau constant étant flasque, on en déduit que l'application $T' \to \mathfrak{N}(T', U)$ est également un faisceau.

Proposition 4. *Soient U une variété semi-complète, T et T' des variétés normales, h un morphisme dominant de T' dans T; les applications h^*: $\mathfrak{M}(T, U) \to \mathfrak{M}(T', U)$ et h^*: $\mathfrak{N}(T, U) \to \mathfrak{N}(T', U)$ sont alors injectives.*

Soit m un élément de $\mathfrak{M}(T, U)$ tel que $h^*(m)=0$; m définit une famille algébrique f de classes de diviseurs de U paramétrée par T. Puisque $h^*(m) = 0$, on a $f \circ h = 0$; on a donc $f(t) = 0$ pour tous les points de la partie dense $h(T')$ de T, d'où $f = 0$ (Théorème 2) et par suite $m = 0$ (Théorème 3). Supposons maintenant que $h^*(m)$ appartienne au sous-groupe $\mathfrak{M}_0(T', U)$; cela signifie que $f \circ h$ est une application constante de T' dans $\mathfrak{G}(U)$; soit $\mathfrak{c}$ sa valeur. L'application constante $t \to \mathfrak{c}$ est une famille algébrique paramétrée par T, définie par un élément $m_0 \in \mathfrak{M}_0(T, U)$; on a $h^*(m) = h^*(m_0)$, d'où $m = m_0$ et par suite $m \in \mathfrak{M}_0(T, U)$. Ceci démontre la Proposition 4.

Remarque. Si T' est une sous-variété ouverte de T et h l'injection canonique $T' \to T$, on peut donner de la Proposition 4 une démonstration qui ne dépend pas de l'hypothèse que U soit semi-complète. Soit $\mathfrak{k}$ un élément de $\mathfrak{G}(T \times U)$ tel que $h^*(\mathfrak{k})$ soit image réciproque d'un élément de $\mathfrak{G}(T')$ par la projection $T' \times U \to T'$. Si $D \in \mathfrak{k}$, il y a une fonction numérique w' sur $T' \times U$ et un diviseur d' de T' tels que le diviseur induit par D sur $T' \times U$ soit de la forme $\operatorname{div} w' + d' \times U$. La fonction w' se prolonge en une fonction w sur $T \times U$; si $D_1 = D - \operatorname{div} w$, on a

$$\operatorname{Supp} D_1 \subset ((T - T') \cup \operatorname{Supp} d') \times U,$$

d'où il résulte que D_1 est de la forme $d_1 \times U$, d_1 étant un diviseur sur T; ceci

montre que h^* est un homomorphisme injectif de $\mathfrak{M}(T, U)$ dans $\mathfrak{M}(T', U)$. Par ailleurs, si $h^*(\mathfrak{f})$ appartient au noyau de l'homomorphisme $\mathfrak{G}(T' \times U) \to \mathfrak{N}(T', U)$, on voit comme ci-dessus qu'il y a un diviseur principal $\operatorname{div} w$ tel que le diviseur induit par $D - \operatorname{div} w$ sur $T' \times U$ soit de la forme $d' \times U + T' \times e$, d' et e étant des diviseurs sur T' et U respectivement; il en résulte que le diviseur induit par $D - \operatorname{div} w - T \times e$ est $d' \times U$, donc que $D - \operatorname{div} w - T \times e$ est de la forme $d \times U$, d étant un diviseur sur T. Ceci montre que h définit une application injective de $\mathfrak{N}(T, U)$ dans $\mathfrak{N}(T', U)$.

On notera que, si g est un morphisme d'une variété U' dans la variété U, g définit un homomorphisme $g^* : \mathfrak{G}(T \times U) \to \mathfrak{G}(T \times U')$ (on désigne encore par g l'application $(t, x') \to (t, g(x'))$ de $T \times U'$ dans $T \times U$). Soient p et p' les projections de $T \times U$ et de $T \times U'$ sur T; il est clair que g^* applique $p^*(\mathfrak{G}(T))$ dans $p'^*(\mathfrak{G}(T))$, donc définit un homomorphisme, encore noté g^*, de $\mathfrak{M}(T, U)$ dans $\mathfrak{M}(T, U')$. Les homomorphismes g^* définissent l'application $U \to \mathfrak{M}(T, U)$ comme foncteur contravariant en son second argument. Si de plus h est un morphisme d'une variété T' dans la variété T, le diagramme

$$\begin{array}{ccc} \mathfrak{M}(T, U) & \xrightarrow{g^*} & \mathfrak{M}(T, U') \\ \Big\downarrow{\scriptstyle h^*} & & \Big\downarrow{\scriptstyle h^*} \\ \mathfrak{M}(T', U) & \xrightarrow{g^*} & \mathfrak{M}(T', U') \end{array}$$

est commutatif; en effet, le morphisme $r : (t', x') \to (h(t'), g(x'))$ définit un homomorphisme de $\mathfrak{G}(T \times U)$ dans $\mathfrak{G}(T' \times U')$ qui applique $p^*(\mathfrak{G}(T))$ dans $p''^*(\mathfrak{G}(T'))$, p'' désignant la projection de $T' \times U'$ sur T' (cela résulte de ce que $p \circ r = h \circ p''$); r définit donc un homomorphisme de $\mathfrak{M}(T, U)$ dans $\mathfrak{M}(T', U')$. Par ailleurs, r est le composé des morphismes $(t', x') \to (h(t'), x')$ et $(t, x') \to (t, g(x'))$, et est aussi le composé des morphismes $(t', x') \to (t', g(x'))$ et $(t', x) \to (h(t'), x)$; notre assertion résulte immédiatement de là. Ce résultat s'exprime en disant que $(T, U) \to \mathfrak{M}(T, U)$ est un bifoncteur, contravariant par rapport à chacun de ses arguments, sur la catégorie des variétés.

De même, si g est un morphisme $U' \to U$, g définit un homomorphisme $g^* : \mathfrak{N}(T, U) \to \mathfrak{N}(T, U')$, et, si h est un morphisme $T' \to T$, le diagramme

$$\begin{array}{ccc} \mathfrak{N}(T, U) & \xrightarrow{g^*} & \mathfrak{N}(T, U') \\ \Big\downarrow{\scriptstyle h^*} & & \Big\downarrow{\scriptstyle h^*} \\ \mathfrak{N}(T', U) & \xrightarrow{g^*} & \mathfrak{N}(T', U') \end{array}$$

est commutatif, comme le lecteur le vérifiera immédiatement; le symbole $\mathfrak{N}$ apparait donc comme un symbole de bifoncteur sur la catégorie des variétés.

Par ailleurs, nous avons déjà observé que, si T et U sont des variétés, l'isomorphisme canonique $U \times T \rightarrow T \times U$ définit un isomorphisme $\sigma_{T,U}$: $\mathfrak{N}(T, U) \rightarrow \mathfrak{N}(U, T)$. Soient g un morphisme d'une variété U' dans la variété U et h un morphisme d'une variété T' dans la variété T; soient r le morphisme $(t', x') \rightarrow (h(t'), g(x'))$ de $T' \times U'$ dans $T \times U$ et r' le morphisme $(x', t) \rightarrow (g(x'), h(t'))$ de $U' \times T'$ dans $U \times T$; r et r' définissent des homomorphismes r^*: $\mathfrak{N}(T, U) \rightarrow \mathfrak{N}(T', U')$, r'^*: $\mathfrak{N}(U, T) \rightarrow \mathfrak{N}(U', T')$. On vérifie immédiatement que le diagramme

$$\begin{tikzcd} \mathfrak{N}(T, U) \arrow[r, "\sigma_{T,U}"] \arrow[d, "r^*"'] & \mathfrak{N}(U, T) \arrow[d, "r'^*"] \\ \mathfrak{N}(T', U') \arrow[r, "\sigma_{T',U'}"] & \mathfrak{N}(U', T') \end{tikzcd}$$

est commutatif.

Proposition 5. *Soient T une sous-variété ouverte de la droite projective $\bar{K}$ et U une variété normale; $\mathfrak{N}(T, U)$ se réduit alors à $\{0\}$.*

Soit U_1 l'ensemble des points simples de U; il existe alors un homomorphisme injectif $\mathfrak{N}(U, T) \rightarrow \mathfrak{N}(U_1, T)$ (cf. la remarque qui suit la Proposition 4); tenant compte des isomorphismes $\mathfrak{N}(U, T) \cong \mathfrak{N}(T, U)$, $\mathfrak{N}(U_1, T) \cong \mathfrak{N}(T, U_1)$, on voit qu'il de démontrer la Proposition 5 dans le cas où U est non-singulière. Supposons qu'il en soit ainsi; $\bar{K} \times U$ est alors non-singulière; l'injection canonique $T \times U \rightarrow \bar{K} \times U$ définit donc une application surjective de $\mathfrak{D}(\bar{K} \times U)$ sur $\mathfrak{D}(T \times U)$ (Corollaire à la Proposition 1, § I), et par suite aussi une application surjective de $\mathfrak{N}(\bar{K}, U)$ sur $\mathfrak{N}(T, U)$. Il suffira donc de montrer que $\mathfrak{N}(\bar{K}, U) = \{0\}$, ou encore que $\mathfrak{N}(U, \bar{K}) = \{0\}$. Comme U est normale et $\bar{K}$ complète, les éléments de $\mathfrak{M}(U, \bar{K})$ sont en correspondence bi-univoque avec les applications algébriques de U dans $\mathfrak{G}(\bar{K})$. Or la structure du groupe $\mathfrak{G}(\bar{K})$ est bien connue: une condition nécessaire et suffisante pour que deux diviseurs sur $\bar{K}$ soient équivalents est qu'ils aient même degré. Si donc on appelle degré d'une classe $\mathfrak{c} \in \mathfrak{G}(\bar{K})$ le degré commun des diviseurs de $\mathfrak{c}$, pour montrer que $\mathfrak{N}(U, \bar{K}) = \{0\}$, il suffira de montrer que, si f est une application algébrique de U dans $\mathfrak{G}(\bar{K})$, le degré de $f(x)$ $(x \in U)$ ne dépend pas de x (car cela entraînera que f est constante). En vertu du caractère connexe de U, il suffira de montrer que tout point de U admet un voisinage U_0 tel que le degré de $f(x)$ reste constant pour $x \in U_0$. Or il y a un voisinage U_0 de x et une famille algébrique f de diviseurs de $\bar{K}$ paramétrée par U_0 tels

que $f(x) = \mathrm{Cl}.\bar{f}(x)$ si $x \in U_0$ (Proposition 2), et le degré de $\bar{f}(x)$ reste constant pour $x \in U_0$ (Proposition 3, § II) ; la Proposition 5 est donc établie.

V. Criteres de rationalité (II).

Proposition 1. *Soient T une variété et (T', h) un revêtement séparable de T. Soit f une application de T dans l'ensemble des classes de diviseurs d'une variété semi-complète U. Si $f \circ h$ est une application algébrique de T' dans $\mathfrak{G}(U)$, il y a une partie ouverte non vide T_0 de T telle que la restriction de f à T_0 soit une application algébrique de T_0 dans $\mathfrak{G}(U)$.*

On peut évidemment supposer T normale. Il existe un revêtement (T'_1, h_1) de T' tel que $(T'_1, h \circ h_1)$ soit revêtement galoisien normal de T; comme $f \circ h \circ h_1$ est une application algébrique de T'_1 dans $\mathfrak{G}(U)$, on voit qu'on peut supposer que (T', h) est galoisien. Nous désignerons par G le groupe des automorphismes du revêtement (T', h) ; nous considérerons G comme opérant aussi sur $T' \times U$. L'application $f \circ h$ est définie par une classe $\mathfrak{k}'$ de diviseurs de $T' \times U$; nous choisirons un point $x_1 \in U$ et un représentant D' de $\mathfrak{k}'$ tel que $T' \times \{x_1\} \not\subset \operatorname{Supp} D'$. Si s est un élément de G, on a $f \circ h \circ s = f \circ h$; il en résulte que $s^*(\mathfrak{k}') - \mathfrak{k}'$ est image réciproque d'un élément de $\mathfrak{G}(T')$ par la projection $T' \times U$ sur T', donc que $s^*(D') - D'$ est de la forme $\operatorname{div} w_s' + d_s' \times U$, où w_s' est une fonction numérique sur $T' \times U$ et d_s' un diviseur sur T'. Désignons par k l'application $t' \to (t', x_1)$ de T' dans $T' \times U$; montrons que l'on peut choisir w_s' de telle manière que w_s' soit composable avec k et que $w_s' \odot k = 1$. Puisque $T' \times \{x_1\} \not\subset \operatorname{Supp} D'$, $k^*(D')$ est défini ; il en est de même de $k^*(d' \times U) = d'$; par suite, $k^*(\operatorname{div} w_s')$ est défini, ce qui montre que w_s' est composable avec k. Soit p' la projection $T' \times U \to T'$; soit $z' = w_s' \odot k \odot p'$, d'où $z' \odot k = w_s' \odot k$; on a $s^*(D') - D' = \operatorname{div} z'^{-1} w_s' + \operatorname{div} z' + d_s' \times U$; or on a $\operatorname{div} z' = \operatorname{div}(w_s' \odot k) \times U$; remplaçant w_s' par $z^{-1} w_s'$ et d_s' par $d_s' + \operatorname{div}(w_s' \odot k)$, on voit qu'on peut supposer que $w_s' \odot k = 1$. La fonction w_s' est alors uniquement determinée. Pour le montrer, nous avons à établir que, si w'' est une fonction numérique sur $T' \times U$ telle que $\operatorname{div} w''$ soit de la forme $d'' \times U$, d'' étant un diviseur sur T', et si $w'' \odot k = 1$, on a $w'' = 1$. Pour tout $t' \in T'$, soit $j'_{t'}$ l'application $x \to (t', x)$ de U dans $T' \times U$; si $t' \notin \operatorname{Supp} d''$, $j_t^*(\operatorname{div} w'')$ est défini et nul, ce qui montre que $w'' \odot j'_{t'}$ est une fonction de diviseur nul sur U, et par suite constante, puisque U est semi-complète. Par ailleurs, (t', x_1) n'est pas dans $\operatorname{Supp} \operatorname{div} w''$, de sorte que w'' est définie en (t', x_1) ; comme $w'' \odot k = 1$, on a $w''(t', x_1) = 1$, d'où $w'' \odot j'_{t'} = 1$. Ceci étant vrai pour tous les points de la partie ouverte non vide $T' - \operatorname{Sup} d''$ de T', il en résulte immédiatement que $w'' = 1$.

Ceci étant, on a, $s, t \in G$, $(st)^*(D') - D' = t^*(s^*(D') - D') + t^*(D') - D'$; comme $((w_s' \odot t)w_t') \odot k = 1$, il résulte de l'assertion d'unicité que nous venons de faire que l'on a $w_{st}' = (w_s' \odot t)w_t'$. Le groupe G opère à droite sur le corps des fonctions numériques sur $T' \times U$ au moyen des applications $w' \rightarrow w' \odot s$; la formule précédente signifie que l'application $w' \rightarrow w_s'$ est un cocycle pour G à valeurs dans le corps des fonctions sur $T' \times U$. Il est bien connu qu'il existe alors une fonction numérique $w' \neq 0$ sur $T' \times U$ telle que $w_s' = w'^{-1}(w' \odot s)$ pour tout $s \in G$. Si donc on pose $D'_1 = D' - \operatorname{div} w'$, on a $s^*(D'_1) - D'_1 = d_s' \times U$. Soit T_1 une partie ouverte non vide de T qui ne rencontre aucun des ensembles $h(\operatorname{Supp} d_s')$, et soit $T_1' = h^{-1}(T_1)$; on peut encore considérer G comme opérant sur $T_1' \times U$. Si D''_1 est le diviseur induit par D'_1 sur $T_1' \times U$, on a $s^*(D''_1) = D''_1$ pour tout $s \in G$. Il existe une partie ouverte non vide T_0 de T_1 telle que (T', h) soit non ramifié en tout point de $h^{-1}(T_0)$. Si h_0 est la restriction de h à $h^{-1}(T_0)$ et r le morphisme $(t', x) \rightarrow (h_0(t'), x)$ de $h^{-1}(T_0) \times U$ dans $T_0 \times U$, $(h^{-1}(T_0) \times U, r)$ est un revêtement galoisien non ramifié de $T_0 \times U$. Si D''_0 est le diviseur induit par D''_1 sur $h^{-1}(T_0) \times U$, il résulte de la Proposition 3, § I que D''_0 se met sous la forme $h^*(D_0)$, D_0 étant un diviseur sur $T_0 \times U$. Soit $\mathfrak{k}_0$ la classe de D_0 dans $\mathfrak{G}(T_0 \times U)$; si i est l'injection canonique de $T_0 \times U$ dans $T \times U$, et h_0 la restriction de h à $h^{-1}(T_0)$, $h_0^*(\mathfrak{k}_0)$ est évidemment égal à $i^*(\mathfrak{k}')$. Si donc f_0 est la famille algébrique de diviseurs de U paramétrée par T_0 définie par $\mathfrak{k}_0$, $f_0 \circ h_0$ est la restriction de $f \circ h$ à $h^{-1}(T_0)$; comme h est surjectif, f_0 est la restriction de f à T_0; cette dernière est donc une famille algébrique.

Proposition 2. *Soient U une variété semi-complète, G un groupe algébrique, H un sous-groupe fermé de G, h l'application canonique de G sur G/H. Soit g un homomorphisme de G dans le groupe des classes de diviseurs d'une variété normale et complète U; supposons que H soit contenu dans le noyau de g et que g soit une famille algébrique paramétrée par G; alors l'application f de G/H dans $\mathfrak{G}(U)$ telle $f \circ h = g$ est une famille algébrique paramétrée par G/H.*

Soient e l'élément neutre de G et H_0 la composante connexe de e dans H. Comme e est simple sur G et H_0, il existe un système $(u_1, \cdots, u_n)$ de variables uniformisantes sur G en e, prenant en ce point la valeur 0, tel que l'idéal de définition en e de la variété H_0 soit engendré par $u_1, \cdots, u_m$, m étant un entier $\leqq n$. L'idéal engendré par $u_{m+1}, \cdots, u_n$ dans l'anneau local de e est l'idéal de définition en e d'une sous-variété fermée T' de G; T' est de dimension $n - m$, e est un point isolé de $T' \cap H$, e est simple sur T' et l'espace tangent à G en e est somme directe des espaces tangents à H_0 et à

T' en e. L'application h induit un morphisme h' de T' dans G/H ; comme e est isolé dans $h'^{-1}(h(e)) = T' \cap H$, on a $\dim h'(T') = \dim T' = n - m = \dim G/H$; h' est donc dominant ; c'est un morphisme de degré fini. Nous allons voir qu'il est séparable. Il suffira évidemment pour cela de démontrer que le morphisme $(t', s) \to h'(t')$ $(s \in H_0)$ de $T' \times H_0$ dans G/H est séparable. Or ce morphisme est composé de l'application $(t', s) \to t's$ de $T \times H_0$ dans G et de l'application h de G sur G/H. Comme h est un morphisme séparable, il suffira de montrer que le morphisme $(t', s) \to t's$ de $T' \times H_0$ dans G est séparable (des considérations de dimension montrent tout de suite que ce morphisme est dominant). Il suffira pour cela de montrer que ce morphisme, soit θ, est non ramifié en (e, e). Or l'image par l'application dérivée $D\theta$ de θ en (e, e) de l'espace tangent à $T' \times H_0$ en (e, e) contient les espace tangents à T' et à H_0 en e, puisque $\theta(t', e) = t'$ $(t' \in T')$, $\theta(e, s) = s$ $(s \in H)$; cette image est donc l'espace tangent à G en e tout entier. Or l'espace tangent à $T' \times H_0$ en (e, e) est de dimension n égale à la dimension de l'espace tangent en e ; $D\theta$ est donc injectif, ce qui démontre notre assertion.

Puisque h' est un morphisme dominant de degré fini, il y a une partie ouverte non vide W_0 de G/H telle que, h'_0 désignant la restriction de h' à $h'^{-1}(W_0) = T'_0$, (T'_0, h'_0) soit un revêtement de W_0 ([2], Proposition 4, chap. IV, §III). L'application $f \circ h'_0$ est la restriction de g à T'_0 et est par suite une application algébrique de T'_0 dans $\mathfrak{G}(U)$. Faisant usage de la Proposition 1, on voit qu'il existe une partie ouverte non vide W_1 de W_0 telle que la restriction de f à W_1 soit une application algébrique de W_1 dans $\mathfrak{G}(U)$. Soit z_1 un point de W_1 ; si z_2 est un point quelconque de G/H, il y a une opération s de G qui transforme z_1 en z_2 ; si $z \in W_1$, on a $f(s \cdot z) = g(s) + f(z)$, comme il résulte du fait que g est un homomorphisme. Il en résulte que la restriction de f au voisinage $s \cdot W_1$ de z_2 est une application algébrique de ce voisinage dans $\mathfrak{G}(U)$. Or G/H est une variété non singulière ; il résulte alors de la Proposition 3, § IV que f est une famille algébrique de classes de diviseurs.

Chapitre II.

I. Construction de la jacobienne. Soit C une courbe normale et complète. Pour tout diviseur d sur C, désignons par $\delta(d)$ le degré de d et par $\lambda(d)$ la dimension de l'espace vectoriel composée des fonctions sur C qui sont multiples de d. De la théorie des courbes, nous n'utiliserons que les résultats suivants : 1) le degré de tout diviseur principal est 0 ; 2) les nombres $\delta(d) + \lambda(d)$, pour tous les diviseurs d de C, forment un ensemble borné

inférieurement; si $-g+1$ est le plus petit de ces nombres, g est un entier $\geqq 0$ qu'on appelle le *genre* de la courbe C.

Soient d un diviseur et V l'espace des fonctions qui sont multiples de d. Si $x_1, \cdots, x_m$ sont des points de C, l'espace V' des fonctions qui sont multiples de $d+\sum_{i=1}^{m} x_i$ est de codimension $\leqq m$ dans V. Procédant par récurrence sur m, on voit qu'il suffit de le montrer dans le cas où $m=1$. Soit alors t une fonction de définition de d en x_1; si $u \in V$, $t^{-1}u$ est définie en x_1, et une condition nécessaire et suffisante pour que u soit multiple de x_1+d est que $(t^{-1}u)(x_1)=0$; ceci démontre notre assertion, $u \rightarrow t^{-1}u(x_1)$ étant une forme linéaire sur V.

Si $\mathfrak{k}$ est une classe de diviseurs de C, tous les diviseurs de $\mathfrak{k}$ ont le même degré qu'on note $\delta(\mathfrak{k})$ et qu'on appelle le *degré de la classe* $\mathfrak{k}$. Par ailleurs, si d et d' sont des diviseurs de $\mathfrak{k}$, on a $\lambda(d)=\lambda(d')$, car, si V et V' sont les espaces de fonctions multiples de d et d' respectivement, et si u est une fonction de diviseur $d'-d$, on a $V'=uV$.

Toute classe $\mathfrak{k}$ de degré $r \geqq g$ contient un diviseur positif. Soit en effet d un diviseur de la classe $\mathfrak{k}$; on a

$$\lambda(-d) \geqq -\delta(-d)+1-g=r+1-g>0,$$

ce qui montre qu'il existe une fonction $u \neq 0$ telle que $\operatorname{div} u+d \geqq 0$.

Pour tout entier $m \leqq 1-g$, il existe un diviseur d de degré m tel que $\lambda(d)+\delta(d)=1-g$. Il suffit en effet de montrer que, si l'assertion est vraie pour m, elle l'est aussi pour $m-1$, et, si $m<1-g$, pour $m+1$. Soit d un diviseur de degré m tel que $\lambda(d)+\delta(d)=1-g$. Si x est un point quelconque de C, on a $\delta(d-1\cdot x)=\delta(d)-1$, $\lambda(d) \geqq \lambda(d-1\cdot x)-1$, et par suite $\lambda(d-1\cdot x)+\delta(d-1\cdot x) \leqq 1-g$; comme on a aussi $\lambda(d-1\cdot x)+\delta(d-1\cdot x) \geqq 1-g$, on voit que l'assertion est vraie pour $m-1$. Supposons maintenant que $m<1-g$; on a alors $\lambda(d)>0$. Soit u une fonction $\neq 0$ qui est multiple de d; il est clair qu'il y a un point $x \in C$ tel que u ne soit pas multiple de $d+1\cdot x$; on a donc alors $\lambda(d+1\cdot x)=\lambda(d)-1$, et par suite $\lambda(d+1\cdot x)+\delta(d+1\cdot x)=1-g$, ce qui montre que l'assertion est vraie pour $m+1$.

On va déduire de là qu'il existe une classe de degré g qui ne contient qu'un seul diviseur $\geqq 0$. Soit d un diviseur de degré $-g$ tel que $\lambda(d)+\delta(d)=1-g$, d'où $\lambda(d)=1$. Soit u_1 une base de l'espace des fonctions multiples de d. Si d' est un diviseur positif de la classe de $-d$, on a $d'=\operatorname{div} u-d$, u étant une fonction numérique; comme d' est $\geqq 0$, u est multiple de d, d'où $u=cu_1$, c étant un élément $\neq 0$ de k. Il en résulte bien que la classe de $-d$ ne contient qu'on diviseur positif.

On appelle *non spéciales* les classes de degré g qui ne contiennent qu'on seul diviseur entier; nous dirons qu'un diviseur entier de degré g est non spécial si sa classes est non spéciale.

Soit S^g la puissance symétrique g-ième de la courbe C, et soit d_g la famille canonique de diviseurs de C paramétrée par S^g. *Nous désignerons par* Ω *l'ensemble des points* $z \in S^g$ *tels que* $d_g(z)$ *soit non spécial; de plus, pour tout* $z \in S^g$, *nous désignerons par* $\mathfrak{d}_g(z)$ *la classe de* $d_g(z)$.

PROPOSITION 1. *L'ensemble* Ω *est ouvert dans* S^g.

Disons qu'on diviseur est spécial s'il n'est pas non spécial. Si $d_g(z)$ est spécial, on a $\lambda(-d_g(z)) > 1$, d'où, si $x \in C$, $\lambda(-d_g(z) + 1 \cdot x) > 0$. Il y a donc un diviseur entier de degré g, donc de la forme $d_g(z')$, avec $z' \in S^g$, qui est dans la classe de $d_g(z)$ et qui est $\geqq 1 \cdot x$. Réciproquement, si, pour tout $x \in C$, $\mathfrak{d}_g(z)$ contient un diviseur entier $\geqq 1 \cdot x$, il est clair que $\mathfrak{d}_g(z)$ est spéciale. Il suffira donc de montrer que, si $x \in C$, l'ensemble B_x des $z \in S^g$ tels que $\mathfrak{d}_g(z)$ contienne un diviseur entier $\geqq 1 \cdot x$ est fermé. Or, l'ensemble A_x des $z' \in S^g$ tels que $d_g(z') - 1 \cdot x \geqq 0$ est fermé (Corollaire 3 au Théorème 1, I, § II). Par ailleurs, l'ensemble H des couples $(z', z) \in S^g \times S^g$ tels que $\mathfrak{d}_g(z') - \mathfrak{d}_g(z) = 0$ est fermé (Théorème 2, I, § IV); B_x est l'image de $H \cap (A_x \times S^g)$ par la projection de $S^g \times S^g$ sur son second facteur, et est par suite fermé puisque S^g est une variété complète.

PROPOSITION 2. *Soit* f *une famille algébrique de classes de diviseurs de degré* g *de* C *paramétrée par une variété* T. *L'ensemble* T_1 *des* $t \in T$ *tels que* $f(t)$ *soit non spécial est ouvert. Supposons que* T *soit normale, et qu'il existe un point* $t_0 \in T$ *tel que* $f(t_0)$ *soit une classe non spéciale et contienne un diviseur entier de degré* g *qui soit somme de* g *points distincts de* C. *Il existe alors un morphisme* φ *de* T_1 *dans* Ω *tel que* $f(t) = \mathfrak{d}_g(\varphi(t))$ *pour tout* $t \in T_1$.

Si $t \in T$, la classe $f(t)$, qui est de degré g, contient un diviseur entier, donc de la forme $d_g(z)$, $z \in S^g$. L'ensemble H des $(t, z) \in T \times S^g$ tels que $f(t) - \mathfrak{d}_g(z) = 0$ est fermé (Théorème 2, I, § IV); la projection de $T \times S^g$ induit une application surjective de H sur T. L'ensemble $H \cap (T \times (S^g - \Omega))$ est fermé; il en est de même de son image par la projection $T \times S^g \to T$, puisque S^g est complète. Or, cette image est $T - T_1$; en effet, pour que $t \in T_1$, il suffit qu'il existe un point $z \in \Omega$ tel que $(t, z) \in H$, et, s'il en est ainsi, z est le seul point de S^g tel que $(t, z) \in H$, comme il résulte du fait que $f(t)$ ne contient qu'un seul diviseur entier. Il résulte de là que T_1 est ouvert. Soit $H_1 = H \cap (T_1 \times S^g)$; H_1 est relativement fermé dans $T_1 \times S^g$, et la projection $T_1 \times S^g \to T_1$ induit une bijection p_1 de H_1 sur T_1. Il en résulte d'abord que,

si $H_1 \neq \emptyset$, H_1 est une sous-variété de $T_1 \times S^g$. Il existe en effet une composante irréductible H_1' de H_1 telle que $p_1(H_1')$ soit dense dans T_1. Mais H_1' est fermé dans $T_1 \times S^g$; S^g étant complète, $p_1(H_1')$ est fermé dans T_1, donc égal à T_1; p_1 étant bijectif, il en résulte que $H_1' = H_1$.

Observons maintenant que l'ensemble des points $z \in S^g$ tels que $d_g(z)$ soit somme de g points distincts est ouvert. Son complémentaire est en effet l'image par l'application canonique $s_g\colon C^g \to S^g$ de l'ensemble des points $(x_1, \cdots, x_g)$ de C^g tels que l'on ait $x_i = x_j$ pour au moins un couple d'indices distincts i et j, ensemble qui est évidemment fermé. Soit Ω_1 l'ensemble des points $z \in \Omega$ tels que $d_g(z)$ soit somme de g points distincts; il est ouvert. Nous supposons que T est normale et que $H_1 \cap (T_1 \times \Omega_1)$ n'est pas vide; nous allons alors montrer que p_1 est un isomorphisme de H_1 sur T_1. L'ensemble $H_1 \cap (T_1 \times \Omega_1)$ est une sous-variété ouverte de H_1; l'ensemble des points en lesquels cette variété est normale est lui-même une sous-variété ouverte H_2 de H_1. Par ailleurs, il existe une sous-variété ouverte T_2 de T_1 et une famille algébrique $\bar{f}$ de diviseurs de C paramétrée par T_2 telles que, pour $t \in T_2$, $f(t)$ soit la classe de $\bar{f}(t)$. Comme p_1 est surjectif, $H_3 = H_2 \cap (T_2 \times S^g)$ est une sous-variété ouverte de H_2. La formule $m(t, z) = d_g(z) - \bar{f}(t)$ $((t, z) \in H_3)$ définit une famille algébrique m de diviseurs de C paramétrée par H_3; on a $m(t, z) \sim 0$ pour tout $(t, z) \in H_3$. Soit M le diviseur de definition de m; comme H_3 est normale et C complète, M est de la forme $\operatorname{div} w + e \times C$, w étant une fonction numérique $H_3 \times C$ et e étant un diviseur sur H_3 (Théorème 3, I, § IV). Soit $H_4 = H_3 - \operatorname{Supp} e$; H_4 est une sous-variété ouverte de H_3, et le diviseur de définition de la restriction m_4 de m à H_4 est principal. Soit (t_1, z_1) un point de H_4; nous allons montrer que p_1 est non ramifié en ce point. Soit Λ un vecteur tangent à H_4 (donc aussi à H_1, ou à $T_1 \times S^g$) en (t_1, z_1) dont l'image par la dérivée de p_1 soit nulle; soit L l'image de Λ par la restriction q_4 de la projection $T_1 \times S^g \to S^g$ à H_4. Pour montrer que $\Lambda = 0$, il suffira de montrer que $L = 0$. Le point z_1 appartient à Ω_1, de sorte que $d_g(z_1)$ est somme de g points distincts de C; la famille d_g est donc infinitésimalement injective en z_1 (I, § III); pour montrer que $L = 0$, il suffira donc de montrer que $< L, d_g > = 0$. Or, si p_4 est la restriction p_1 à H_4, on a $m_4 = d_g \circ q_4 - f \circ p_4$; comme l'image de Λ par la dérivée de p_4 est nulle, on a $< \Lambda, f \circ p_4 > = 0$ (Lemme 1, I, § III), d'où $< \Lambda, m_4 > = < \Lambda, d_g \circ p_4 > = < L, d_g >$. Puisque le diviseur de définition de m_4 est principal, le diviseur additif $< \Lambda, m_4 >$ est principal; il est représenté par une fonction numérique s sur C, que l'on peut supposer $\neq 0$ (car $< \Lambda, m_4 >$ est aussi représenté par $s + 1$). Par ailleurs, d_g est une famille de diviseurs $\geqq 0$; son diviseur de définition est donc $\geqq 0$. Le diviseur additif $< d_g, L >$ est représenté par s;

si donc u est une fonction de définition de $d_g(z_1)$ en un point x quelconque de C, su est définie en x (Lemme 2, I, § III). Il en résulte que $\operatorname{div} s + d_g(z_1) \geqq 0$; or ce diviseur appartient à $\mathfrak{d}_g(z_1)$; mais, comme $z_1 \in \Omega$, $\mathfrak{d}_g(z_1)$ ne contient qu'un seul diviseur $\geqq 0$, à savoir $d_g(z_1)$; on a donc $\operatorname{div} s = 0$. Comme C est complète, s est une constante; comme s est une fonction de définition du diviseur additif $< d_g, L >$, ce dernier est nul, d'où $L = 0$.

Comme p_1 est un morphisme dominant non ramifié en (t_1, z_1), ce morphisme est séparable ([2], Corollaire 2 à la Proposition 3, II, chap. VI). Comme p_1 est injectif, il est aussi radiciel; p_1 est donc birationnel. Comme T_1 est une variété normale, il résulte du théorème principal de Zariski que p_1 est un isomorphisme de H_1 sur T_1. En composant l'isomorphisme réciproque de p_1 avec la restriction à H_1 de la projection $T \times S^g \to S^g$, on obtient un morphisme φ de T dans S^g; il est clair que $f(t) = \mathfrak{d}_g(\varphi(t))$ pour $t \in T_1$.

Nous désignerons par $\mathfrak{G}_0(C)$ (resp. $\mathfrak{G}_g(C)$) l'ensemble des classes de diviseurs de degré 0 (resp. g) de C. Si $\mathfrak{c} \in \mathfrak{G}_g(C)$, nous désignerons par $\theta_\mathfrak{c}$ l'application $z \to \mathfrak{d}_g(z) - \mathfrak{c}$ de S^g dans $\mathfrak{G}_0(C)$, par $V(\mathfrak{c})$ l'ensemble $\theta_\mathfrak{c}(\Omega)$ et par $\theta_\mathfrak{c}'$ la restriction de $\theta_\mathfrak{c}$ à Ω. Si $z \in \Omega$, $\theta_\mathfrak{c}^{-1}(\theta_\mathfrak{c}(z))$ se compose du seul point z, puisque $\mathfrak{d}_g(z)$ ne contient qu'un seul diviseur entier (et que l'application $d_g \colon S^g \to \mathfrak{D}(C)$ est injective). Et particulier, $\theta_\mathfrak{c}'$ est une bijection de Ω sur $V(\mathfrak{c})$; elle permet de transporter à $V(\mathfrak{c})$ la structure de variété de Ω. L'ensemble $\mathfrak{G}_0(C)$ est la réunion des ensembles $V(\mathfrak{c})$ pour tous les $\mathfrak{c} \in \mathfrak{G}_g(C)$; en effet, si $\mathfrak{k}$ est un élément de $\mathfrak{G}_0(C)$, $\mathfrak{c} \to \mathfrak{k} + \mathfrak{c}$ est une permutation de l'ensemble $\mathfrak{G}_g(C)$, de sorte que, si $z_1 \in \Omega$, il existe toujours un $\mathfrak{c} \in \mathfrak{G}_g(C)$ tel que $\mathfrak{k} + \mathfrak{c} = \mathfrak{d}_g(z_1)$, d'où $\mathfrak{k} \in V(\mathfrak{c})$. On voit en même temps que, si z_1 est un point quelconque de S^g, $\mathfrak{c} \to \mathfrak{d}_g(z_1) - \mathfrak{c}$ est une bijection de $\mathfrak{G}_g(C)$ sur $\mathfrak{G}_0(C)$; or toute classe de degré g contient un diviseur entier, donc de la forme $\mathfrak{d}_g(z)$, $z \in S^g$; comme $\mathfrak{k} \to -\mathfrak{k}$ est une permutation de $\mathfrak{G}_0(C)$, on voit que, pour tout $z_1 \in S^g$, l'application $z \to \mathfrak{d}_g(z_1)$ est une surjection de S^g sur $\mathfrak{G}_g(C)$. Nous allons maintenant voir qu'il existe sur $\mathfrak{G}_0(C)$ une structure de variété (et, naturellement, une seule) telle que, pour tout $\mathfrak{c} \in \mathfrak{G}_g(C)$, $V(\mathfrak{c})$ soit une sous-variété ouverte de cette variété. Pour montrer qu'il en est ainsi, il suffit de montrer que les conditions suivantes sont satisfaites: 1) si $\mathfrak{c}, \mathfrak{c}' \in \mathfrak{G}_g(C)$, $V(\mathfrak{c}) \cap V(\mathfrak{c}')$ est ouvert dans $V(\mathfrak{c})$ et $V(\mathfrak{c}')$ et les structures de variété induites sur cet ensemble par celles de $V(\mathfrak{c})$ et de $V(\mathfrak{c}')$ sont identiques; de plus, l'ensemble des points de la forme $(\mathfrak{k}, \mathfrak{k})$, $\mathfrak{k} \in V(\mathfrak{c}) \cap V(\mathfrak{c}')$, est fermé dans $V(\mathfrak{c}) \times V(\mathfrak{c}')$; 2) l'ensemble $\mathfrak{G}_0(C)$ est la réunion d'un nombre fini des ensembles $V(\mathfrak{c})$.

Vérifions d'abord la condition 1; $V(\mathfrak{c}) \cap V(\mathfrak{c}')$ est l'image par $\theta_\mathfrak{c}$ (resp. $\theta_{\mathfrak{c}'}$) d'une partie Ω' (resp. Ω'') de Ω. L'ensemble Ω' est l'ensemble des $z' \in \Omega$

tels que $\mathfrak{d}_g(z') - \mathfrak{c} + \mathfrak{c}'$ soit non spéciale. Or, l'application

$$f: z' \to \mathfrak{d}_g(z') - \mathfrak{c} + \mathfrak{c}' \qquad (z' \in \Omega)$$

est une famille algébrique de classes de diviseurs de C paramétrée par Ω; ceci montre que Ω' est ouvert (Proposition 2) ; on voit de même que Ω'' est ouvert; $V(\mathfrak{c}) \cap V(\mathfrak{c}')$ est donc ouvert dans $V(\mathfrak{c})$ et $V(\mathfrak{c}')$. Montrons qu'il existe un point $z_0 \in \Omega$ tel que $f(z_0)$ soit somme de g points distincts de C. Considérons pour cela l'ensemble H des points $(z', z'') \in \Omega \times S^g$ tels que l'on ait $\mathfrak{d}_g(z') = \mathfrak{d}_g(z'') + \mathfrak{c} - \mathfrak{c}'$. Cet ensemble est fermé; si p_1 et p_2 sont les restrictions à H des projections de $\Omega \times S^g$ sur son premier et son second facteur, p_1 est surjectif (car, si $z' \in \Omega$, $\mathfrak{d}_g(z')$ peut se mettre sous la forme $\mathfrak{d}_g(z'') + \mathfrak{c} - \mathfrak{c}'$, avec un $z'' \in S^g$) et p_2 est injectif, car, si $\mathfrak{d}_g(z'') + \mathfrak{c} - \mathfrak{c}'$ est non spéciale, cette classe ne contient qu'un diviseur entier. On déduit de la première assertion que $\dim H \geqq g$, de la seconde que $\dim H = \dim p_2(H) \leqq g$; on a donc $\dim p_2(H) = \dim H = g$, et $p_2(H)$ est dense dans S^g, donc rencontre l'ensemble Ω_1 des points $z'' \in \Omega$ tels que $d_g(z'')$ soit somme de g points distincts. Il résulte alors de la Proposition 2 qu'il existe un morphisme ω de Ω' dans Ω'' tel que $f(z) = \mathfrak{d}_g(\omega(z))$ $(z \in \Omega')$. On voir de même qu'il existe un morphisme ω' de Ω'' dans Ω' tel que $\mathfrak{d}_g(z'') + \mathfrak{c}' - \mathfrak{c} = \mathfrak{d}_g(\omega'(z''))$ $(z'' \in \Omega'')$. Il s'en suit que ω et ω' sont des isomorphismes réciproques l'un de l'autre. Comme ω est l'application définie par la condition que $\theta_{\mathfrak{c}'} \circ \omega = \theta_{\mathfrak{c}}$, on voit que $V(\mathfrak{c})$ et $V(\mathfrak{c}')$ induisent la même structure de variété sur leur intersection. L'ensemble des $(\mathfrak{k}, \mathfrak{k})$, $\mathfrak{k} \in V(\mathfrak{c}) \times V(\mathfrak{c}')$, est l'image part l'application $(z', z'') \to (\theta_{\mathfrak{c}}(z'), \theta_{\mathfrak{c}'}(z''))$ de l'ensemble $H \cap (\Omega \times \Omega)$, qui est fermé dans $\Omega \times \Omega$; il est donc fermé dans $V(\mathfrak{c}) \times V(\mathfrak{c}')$.

Il reste à montrer que $\mathfrak{G}_0(C)$ est la réunion d'un nombre fini des variétés $V(\mathfrak{c})$. Soit $\mathfrak{c}_0$ une classe quelconque de degré g; $\mathfrak{G}_0(C)$ est alors l'image de S^g par l'application $\theta_{\mathfrak{c}_0}$: $z \to \mathfrak{d}_g(z) - \mathfrak{c}_0$; pour tout $\mathfrak{c} \in \mathfrak{G}_g(C)$, $V(\mathfrak{c})$ est l'image par $\theta_{\mathfrak{c}_0}$ de l'ensemble $W(\mathfrak{c})$ des $z \in S^g$ tels que $\mathfrak{d}_g(z) + \mathfrak{c} - \mathfrak{c}_0$ soit non spéciale, ensemble qui est ouvert en vertu de la Proposition 2. La variété S^g est la réunion des ensembles ouverts $W(\mathfrak{c})$; elle est donc la réunion d'un nombre fini d'entre eux, ce qui démontre notre assertion.

La variété J dont l'ensemble de points est $\mathfrak{G}_0(C)$ et dont les $V(\mathfrak{c})$ sont des sous-variétés ouvertes s'appelle la *jacobienne* de C.

THÉORÈME 1. *Soient C une courbe normale et complète, T une variété normale, t_0 un point de T, f une application de T dans $\mathfrak{G}(C)$, J la jacobienne de C. Pour que f soit une famille algébrique de classes de diviseurs de C paramétrée par T, il faut et suffit que l'application $t \to f(t) - f(t_0)$ soit un morphisme de T dans J.*

Supposons d'abord que f soit algébrique. On sait que le degré de $f(t)$ est indépendant de t (Proposition 3, I, II) ; il en résulte que $f(t) - f(t_0) \in J$ pour tout $t \in T$. Soit t_1 un point de T. Soient $\mathfrak{k}_0$ un élément quelconque de $\mathfrak{G}_0(C)$ et $\mathfrak{c}_1$ un élément de $\mathfrak{G}_g(C)$ tel que $f(t_1) - f(t_0) + \mathfrak{k}_0 + \mathfrak{c}_1$ contienne un diviseur entier non spécial qui soit la somme de g points distincts de C. Il résulte alors de la Proposition 2 que l'ensemble T_1 des $t \in T$ tels que $f(t) - f(t_0) + \mathfrak{k}_0 + \mathfrak{c}_1$ soit non spéciale est un voisinage ouvert de t_1 et qu'il existe un morphisme φ de T_1 dans Ω tel que

$$f(t) - f(t_0) + \mathfrak{k}_0 + \mathfrak{c}_1 = \mathfrak{d}_g(\varphi(t))$$

pour $t \in T_1$; la restriction à T_1 de l'application $t \to f(t) - f(t_0)$ est donc l'application $t \to \mathfrak{d}_g(\varphi(t)) - \mathfrak{c}_1$ qui est un morphisme de T_1 dans J en vertu de la construction de la jacobienne. Il résulte de là que l'application $t \to f(t) - f(t_0) + \mathfrak{k}_0$, qui coincide au voisinage de chaque point avec un morphisme d'un voisinage de ce point dans J, est un morphisme de T dans J ; prenant en particulier $\mathfrak{k}_0 = 0$, on voit que $t \to f(t) - f(t_0)$ est un morphisme de T dans J.

Supposons réciproquement que $t \to f(t) - f(t_0)$ soit un morphisme de T dans J. Pour montrer que f est algébrique, il suffira évidemment de montrer que l'application identique i de J sur $\mathfrak{G}_0(C)$ est une famille algébrique de classes de diviseurs de C. Si $\mathfrak{c} \in \mathfrak{G}_g(C)$, la restriction $i_\mathfrak{c}$ de i à la sous-variété $V(\mathfrak{c})$ de J définie ci-dessus est une famille algébrique, car on l'obtient en composant l'isomorphisme $\theta_\mathfrak{c}^{-1}$ de $V(\mathfrak{c})$ sur Ω avec la famille algébrique $z \to \mathfrak{d}_g(z) - \mathfrak{c}$. Tenant compte de ce qui a été dit dans la première partie de la démonstration, on voit que, pour tout $\mathfrak{k}_0 \in \mathfrak{G}_0(C)$, l'application $\mathfrak{k} \to \mathfrak{k} + \mathfrak{k}_0$ ($\mathfrak{k} \in V(\mathfrak{c})$) est un morphisme de $V(\mathfrak{c})$ dans J. Ceci étant vrai pour tout $\mathfrak{c}$, on voit que l'application $\mathfrak{k} \to \mathfrak{k} + \mathfrak{k}_0$ est un morphisme de J dans J ; c'est même un automorphisme de J, l'application $\mathfrak{k} \to \mathfrak{k} - \mathfrak{k}_0$ étant également un morphisme. La variété J, qui admet un groupe transitif d'automorphismes, est donc une variété non singulière. Faisant usage de la Propostition 3, I, § IV, on déduit alors du fait que les $i_\mathfrak{c}$ sont des familles algébriques de classes de diviseurs qu'il en est de même de f, ce qui démontre le Théorème 1.

COROLLAIRE 1. *La variété J, munie de la structure de groupe de $\mathfrak{G}_0(C)$, est une variété abélienne.*

L'application $(\mathfrak{k}, \mathfrak{k}') \to \mathfrak{k} - \mathfrak{k}'$ de $J \times J$ dans J est une famille algébrique de classes de diviseurs de C, puisque $\mathfrak{k} \to \mathfrak{k}$ en est une. C'est donc un morphisme de $J \times J$ dans J, ce qui montre que J est un groupe algébrique. Soit z_1 un point de S^g ; l'application $z \to \mathfrak{d}_g(z) - \mathfrak{d}_g(z_1)$ ($z \in S^g$) est une

famille algébrique de classes de diviseurs; c'est donc un morphisme de S^g dans J. On sait par ailleurs que ce morphisme est surjectif. Comme S^g est complète, il en est de même de J.

COROLLAIRE 2. *Pour tout $x \in C$, soit $\mathfrak{x}(x)$ la classe du diviseur $1 \cdot x$; si $x_0 \in C$, l'application $x \rightarrow \mathfrak{x}(x) - \mathfrak{x}(x_0)$ est un morphisme de C dans J.*

Cela résulte immédiatement du Théorème 1. Une application de C dans J définie de cette manière est appelée *canonique.*

II. Familles de classes de diviseurs parametrées par une courbe. Nous désignerons par C une courbe complète et normale. Pour tout $r > 0$, nous désignerons par S^r la puissance symétrique r-ième de C, par d_r la famille canonique de diviseurs de C paramétrée par S^r, et, pour $z \in S^r$, par $\mathfrak{d}^r(z)$ la classe de $d^r(z)$.

PROPOSITION 1. *Soit $\mathfrak{a}$ une classe de diviseurs de C. Si l'ensemble E des points $z \in S^r$ tels que $\mathfrak{d}_r(z) = \mathfrak{a}$ n'est pas vide, c'est une sous-variété fermée de S^r isomorphe à un espace projectif.*

Supposons $E \neq \emptyset$, et soit z_1 un point de cet ensemble; soit V l'espace vectoriel des fonctions multiples de $-d_r(z_1)$; désignons par $\zeta \rightarrow \operatorname{div} \zeta$ le système linéaire de diviseurs de C paramétré par l'espace projectif $\mathfrak{P}(V)$ associé à V. Si $\zeta \in \mathfrak{P}(V)$, il y a un point $f(\zeta)$et un seul de S^r tel que $d_r(f(\zeta)) = \operatorname{div} \zeta + d_r(z_1)$; il est clair que f est une bijection de $\mathfrak{P}(V)$ sur E. Nous allons montrer que c'est un morphisme de $\mathfrak{P}(V)$ dans S^r. Dans le cas où $\mathfrak{a}$ contient un diviseur qui est somme de r points distincts de C, cela résulte de la Proposition 2, I, § III et du fait que d_r est infinitésimalement injective en tout point z tel que $d_r(z)$ soit somme de r points distincts. Pour établir notre assertion dans le cas général, nous observerons qu'il existe un diviseur entier b que la classe de $d_r(z_1) + b$ contienne un diviseur qui soit la somme de $r + r'$ points distincts (où r' est le degré de b). En effet, soit a un diviseur de degré $r + g$ qui soit somme de $r + g$ points distincts; l'espace des fonctions qui sont multiples de $-a$ est de dimension $\geqq r + 1$, d'où on déduit qu'il existe une fonction $\neq 0$ multiple de $-a + d_r(z_1)$; si $-a + d_r(z_1) + b$ est le diviseur de cette fonction, b possède la propriété requise (on peut donc prendre $r' = g$; nous supposons qu'il en est ainsi). Soit Σ l'ensemble des points $t \in S^{r+g}$ tels que $d_{r+g}(t)$ soit multiple de b; il y a une application bijective σ de Σ sur S^r telle que $d_r(\sigma(t)) = d_{r+g}(t) - b$ $(t \in \Sigma)$. Montrons que Σ est une sous-variété fermée de S^{r+g} et que σ est un morphisme. L'ensemble Σ est fermé en vertu du Corollaire 3 au Théorème 1, § I; il y au moins une composante

irréductible Σ' de Σ telle que $\sigma(\Sigma')$ soit dense dans S^r. L'application $t \to d_{r+g}(t) - b$ induit une famille algébrique de diviseurs de C paramétrée par Σ'; par ailleurs, les points z tels que $d_r(z)$ soit somme de r points distincts forment une partie ouverte non vide de S^r, de sorte que $\sigma(\Sigma')$ rencontre cet ensemble. Faisant usage du résultat cité plus haut, on voit que la restriction de σ à Σ' est un morphisme de Σ' dans S^r. Comme Σ' est fermé dans S^{r+g}, c'est une variété complète; $\sigma(\Sigma')$ est donc fermé, d'où $\sigma(\Sigma') = S^r$ et $\Sigma' = \Sigma$ puisque σ est injectif. Ceci étant, soit V' l'espace des fonctions qui sont multiples de $-d_r(z_1) - b$; soit $\zeta' \to \operatorname{div} \zeta'$ le système linéaire paramétrée par $\mathfrak{P}(V')$. Il résulte de ce que nous avons dit que l'application f' de $\mathfrak{P}(V')$ dans S^{r+g} définie par la condition que $d_{r+g}(f'(\zeta)) = \operatorname{div} \zeta + d_r(z_1) + b$ $(\zeta \in \mathfrak{P}(V'))$ est un morphisme. Par ailleurs, on a $V \subset V'$, de sorte que $\mathfrak{P}(V)$ est une sous-variété de $\mathfrak{P}(V')$. Il est clair que $f'(\mathfrak{P}(V)) \subset \Sigma$ et que $f(\zeta) = \sigma(f'(\zeta))$ si $\zeta \in \mathfrak{P}(V)$; f est donc bien un morphisme.

Il résulte de là que E est une sous-variété fermée de S^r. Soit g l'application réciproque de f; montrons que g est un morphisme de E dans $\mathfrak{P}(V)$. Si $z \in E$, $g(z)$ est défini par la condition que $\operatorname{div} g(z) = d_r(z) - d_r(z_1)$; comme la famille $\zeta \to \operatorname{div} \zeta$ est injective et infinitésimalement injective, le fait que g soit un morphisme résulte de la Proposition 2, I, § III. L'application f est donc un isomorphisme de $\mathfrak{P}(V)$ sur E.

Proposition 2. *Soit W une sous-variété ouverte de S^r, et soit f une famille algébrique de classes de diviseurs d'une variété normale U paramétrée par W. Si z, z' sont des points de W tels que $\mathfrak{d}_r(z) = \mathfrak{d}_r(z')$, on a $f(z) = f(z')$.*

Il résulte immédiatement de la Proposition 1 qu'il y a une sous-variété E de S^r isomorphe à la droite projective passant par z et z' (sauf si $z = z'$, auquel cas le résultat est évident). Soit E_0 l'ensemble $E \cap W$; E_0 est isomorphe à une sous-variété ouverte de la droite projective, et la restriction f_0 de f à E_0 est une famille algébrique de classes de diviseurs de U paramétrée par E_0. Or on a $\mathfrak{R}(E_0, U) = \{0\}$ (Proposition 5, I, § IV); f_0 est donc constante, ce qui démontre la Proposition 2, puisque z et z' appartiennent à E_0.

Proposition 3. *Soient J la jacobienne de C et χ une application canonique de C dans J. Soient C_1 une sous-variété ouverte de C et f une famille algébrique de classes de diviseurs d'une variété normale et semi-complète U paramétrée par C_1. Il y a alors une famille algébrique g de classes de diviseurs de U paramétrée par J telle que $g(\chi(x)) = f(x)$ pour tout $x \in C_1$; si $\mathfrak{k}_0$ est un point de J, l'application $\mathfrak{k} \to g(\mathfrak{k}) - g(\mathfrak{k}_0)$ est un homomorphisme du groupe J dans $\mathfrak{G}(U)$.*

Montrons que, si C_2 est une sous-variété ouverte de C_1 et si la proposition est vraie pour la restriction f_2 de f à C_2, elle est vraie pour f. Soit g une application algébrique de J dans $\mathfrak{G}(U)$ telle que $g(\chi(x)) = f_2(x)$ pour $x \in C_2$; l'application $x \to g(\chi(x))$ $(x \in C_1)$ est une famille algébrique de classes de diviseurs de U paramétrée par C_1 et qui coincide avec f sur C_2; cette famille est donc identique à f (Théorème 2, I, § IV). On peut donc supposer sans restriction de généralité qu'il existe une famille algébrique $\bar{f}$ de diviseurs de U paramétrée par C_1 telle que $f(x) = \mathrm{Cl}.\bar{f}(x)$ pour $x \in C_1$. Soit alors r un entier $\geqq 0$; soit W_r l'ensemble des points z de S^r tels que $d_r(z)$ soit la somme de r points de C_1; c'est une sous-variété ouverte de S^r qui s'identifie à la puissance symétrique r-ième de C_1. Il existe donc une famille algébrique $\bar{h}_r$ de diviseurs de U paramétrée par W_r telle que $\bar{h}_r(z) = \bar{f}(x_1) + \cdots + \bar{f}(x_r)$ si $d_r(z) = x_1 + \cdots + x_r$, avec $x_i \in C_1$ $(1 \leqq i \leqq r)$. Si $z \in W_r$, soit $h_r(z)$ la classe de $\bar{h}_r(z)$; il résulte de la Proposition 2 que la condition $d_r(z) \sim d_r(z')$ $(z, z' \in C_1)$ entraîne $h_r(z) = h_r(z')$.

Soit Ω_1 l'intersection de W_g avec l'ensemble des points $z \in S^g$ tels que $d_g(z)$ soit non spécial. Soit z_1 un point quelconque de W_g; rappelons que l'application $z \to \mathfrak{d}_g(z) - \mathfrak{d}_g(z_1)$ $(z \in \Omega_1)$ est un isomorphisme de la sous-variété ouverte Ω_1 de S^g sur une sous-variété ouverte $V(z_1)$ de J. Il y a donc une application algébrique g'_{z_1} *de* $V(z_1)$ dans $\mathfrak{G}(U)$ telle que

$$g'_{z_1}(\mathfrak{d}_g(z) - \mathfrak{d}_g(z_1)) = h_g(z) - h_g(z_1)$$

pour tout $z \in \Omega_1$. Montrons que, si z_1, z_1' sont des points de W_g, g'_{z_1} coincide avec $g'_{z_1'}$ dans $V(z_1) \cap V(z_1')$. Soient z et z' des points de Ω tels que $\mathfrak{d}_g(z) - \mathfrak{d}_g(z_1) = \mathfrak{d}_g(z') - \mathfrak{d}_g(z_1')$, d'où $\mathfrak{d}_g(z) + \mathfrak{d}_g(z_1') = \mathfrak{d}_g(z') + \mathfrak{d}_g(z_1)$. Il y a des points t, t' de W_{2g} tels que $d_{2g}(t) = d_g(z) + d_g(z_1')$, $d_{2g}(t') = d_g(z') + d_g(z_1)$, et on a $d_{2g}(t) \sim d_{2g}(t')$; il en résulte que $h_{2g}(t) = h_{2g}(t')$. Mais il est clair que $h_{2g}(t) = h_g(z) + h_g(z_1')$, $h_{2g}(t') = h_g(z') + h_g(z_1)$; notre assertion est donc établie.

Tout point de J peut se mettre sous la forme $\mathfrak{d}_g(z) - \mathfrak{d}_g(z_1)$ avec z et z_1 dans Ω_1 (Corollaire 3 au Théorème 1, § I). Il y a donc une application g' de J dans $\mathfrak{G}(U)$ qui prolonge toutes les applications $g_{z_1'}$. Comme J est une variété non singulière, g' est une famille algébrique de classes de diviseurs (Propositions 3, I, § IV). Montrons que c'est un homomorphisme de J dans $\mathfrak{G}(U)$. Soient z_1 un point de W_g et $\mathfrak{k}$, $\mathfrak{k}'$ des éléments de $V(z_1)$; écrivons $\mathfrak{k} = \mathfrak{d}_g(z) - \mathfrak{d}_g(z_1)$, $\mathfrak{k}' = \mathfrak{d}_g(z') - \mathfrak{d}_g(z_1)$, avec $z, z' \in \Omega_1$; on a $\mathfrak{k} - \mathfrak{k}' = \mathfrak{d}_g(z) - \mathfrak{d}_g(z')$, et par suite

$$g'(\mathfrak{k}) = h_g(z) - h_g(z_1), \qquad g'(\mathfrak{k}') = h_g(z') - h_g(z_1),$$
$$g'(\mathfrak{k} - \mathfrak{k}') = h_g(z) - h_g(z') = g'(\mathfrak{k}) - g'(\mathfrak{k}') ;$$

les applications $(\mathfrak{k}, \mathfrak{k}') \to g'(\mathfrak{k} - \mathfrak{k}')$, $(\mathfrak{k}, \mathfrak{k}') \to g'(\mathfrak{k}) - g'(\mathfrak{k}')$ sont des applications algébriques de $J \times J$ dans $\mathfrak{G}(U)$ qui coincident sur l'ensemble ouvert non vide $V(z_1) \times V(z_1)$ et qui sont par suite égales, ce qui démontre que g' est un homomorphisme.

Si $z_1 \in W_g$, l'application $z \to \mathfrak{d}_g(z) - \mathfrak{d}_g(z_1)$ est un morphisme de S^g tout entier dans J; les applications $z \to g'(\mathfrak{d}_g(z) - \mathfrak{d}_g(z_1))$ et $z \to h_g(z) - h_g(z_1)$ $(z \in W_g)$ sont des familles algébriques de classes de diviseurs paramétrées par W_g qui coincident sur Ω_1 et qui sont par suite égales. Si $x \in C$, désignons par $\mathfrak{x}(x)$ la classe de $1 \cdot x$; si $g > 0$, soit m un diviseur entier de degré $g-1$ de support contenu dans C_1; écrivons $m = d_{g-1}(c)$, $c \in S^{g-1}$. Soit x_1 un point de C; prenons pour z_1 le point de W_g tel que $d_g(z_1) = 1 \cdot x_1 + m$; si $x \in C_1$, soit z le point de W_g tel que $d_g(z) = 1 \cdot x + m$. On a

$$\mathfrak{x}(x) - \mathfrak{x}(x_1) = \mathfrak{d}_g(z) - \mathfrak{d}_g(z_1),$$

d'où

$$\begin{aligned} g'(\mathfrak{x}(x) - \mathfrak{x}(x_1)) &= h_g(z) - h_g(z_1) \\ &= (f(x) + h_{g-1}(c)) - (f(x_1) + h_{g-1}(c)) = f(x) - f(x_1). \end{aligned}$$

Par ailleurs, il y a un point $x_0 \in C$ tel que $\chi(x) = \mathfrak{x}(x) - \mathfrak{x}(x_0)$; si nous posons $g(\mathfrak{k}) = g'(\mathfrak{k}) + f(x_1) - g'(\mathfrak{x}(x_1) - \mathfrak{x}(x_0))$, on a $g(\chi(x)) = f(x)$ si $x \in C_1$, ce qui établit l'existence de g dans le cas où C est de genre > 0. Si C est de genre 0, on a $J = \{0\}$ et f est constante, car, si $x, x' \in C_1$, on a $1 \cdot x \sim 1 \cdot x'$, d'où $f(x) = h_1(x) = h_1(x') = f(x')$. De plus, comme g ne diffère de g' que par une constante, on a $g(\mathfrak{k}) - g(\mathfrak{k}_0) = g'(\mathfrak{k})$, de sorte que l'application $\mathfrak{k} \to g(\mathfrak{k}) - g(\mathfrak{k}_0)$ est un homomorphisme.

Chapitre III.

I. Definition de la notion de variete de Picard. Soient U une variété et G un groupe algébrique. Nous appellerons *homomorphisme algébrique* de G dans $\mathfrak{G}(U)$ un homomorphisme du groupe G dans $\mathfrak{G}(U)$ qui est en même temps une application algébrique de G dans $\mathfrak{G}(U)$.

On dit qu'un couple (P, π) formé d'un groupe algébrique P et d'un homomorphisme algébrique π de P dans $\mathfrak{G}(U)$ est une *variété de Picard* de U si la condition suivante est satisfaite: pour tout groupe algébrique G et tout homomorphisme algébrique γ de G dans $\mathfrak{G}(U)$, il existe un homomorphisme φ et un seul de G dans P tel que $\gamma = \pi \circ \varphi$ [quand nous parlons d'homomorphismes de groupes algébriques, il s'agit bien entendu d'homomorphismes rationnels].

Soient U et U' des variétés qui admettent des variétés de Picard (P,π) et (P',π'), et soit f un morphisme de U' dans U. Alors $x \rightarrow f^*(\pi(x))$ est un homomorphisme algébrique de P dans $\mathfrak{G}(U')$; il existe donc un homomorphisme φ_f et un seul de P dans P' tel que $f^*(\pi(x)) = \pi'(\varphi_f(x))$ pour tout $x \in P$. Si U'' est une troisiéme variété qui admet une variété de Picard (P'',π''), et si f'' est un morphisme de U'' dans U', il est clair que l'on a $\varphi_{f\circ f'} = \varphi_{f'} \circ \varphi_f$; il en résulte en particulier que, si (P,π) et (P',π') sont des variétés de Picard d'une même variété U, il y a un isomorphisme φ et un seul de P sur P' tel que $\pi = \pi' \circ \varphi$.

PROPOSITION 1. *Si une variété normale et semi-complète U admet une variété de Picard (P,π), l'homomorphisme π est injectif.*

Soit en effet N son noyau. Comme π est une application algébrique, N est un sous-groupe fermé de P (Théorème 2, I, § IV) ; soit ω l'application canonique de P sur P/N. Il y a alors un homomorphisme π^* du groupe P/N dans $\mathfrak{G}(U)$ tel que $\pi^* \circ \omega = \pi$, et π^* est algébrique (Proposition 2, I, § V). Il y a donc un homomorphisme φ de P/N dans P tel que $\pi^* = \pi \circ \varphi$, d'où $\pi = \pi \circ \varphi \circ \omega$. It en résulte que $\varphi \circ \omega$ est l'automorphisme identique de P, de sorte que N se réduit à son élément neutre.

COROLLAIRE. *Une variété de Picard est toujours un groupe commutatif.*

PROPOSITION 2. *Soit U une variété normale et semi-complète. Pour que U admette une variété de Picard, il faut et suffit que la condition suivante soit satisfaite: si (G_n,γ_n) est une suite de couples composés chacun d'un groupe algébrique G_n et d'un homomorphisme injectif algébrique $\gamma_n\colon G_n \rightarrow \mathfrak{G}(U)$, et si ω_n est, pour tout n, un homomorphisme de G_n dans G_{n+1} tel que $\gamma_{n+1} \circ \omega_n = \gamma_n$, alors il existe un entier n_0 tel que, pour tout $n \geqq n_0$, γ_n soit un isomorphisme.*

Supposons d'abord que U admette une variété de Picard (G,γ). Il existe alors pour tout n un homomorphisme θ_n de G_n dans G tel que $\gamma_n = \gamma \circ \theta_n$; comme γ_n est injectif, il en est de même de θ_n. On a $\gamma \circ \theta_{n+1} \circ \omega_n = \gamma_{n+1} \circ \omega_n = \gamma_n$; comme θ_n est uniquement determiné par la condition que nous lui avons imposée, on a $\theta_{n+1} \circ \omega_n = \theta_n$. Les sous-groupes fermés $\theta_n(G_n)$ de G forment donc une suite croissante, ce qui montre qu'ils sont tous égaux à partir d'un certain rang n_1; soit G' leur valeur commune. Pour $n \geqq n_1$, ω_n est un morphisme bijectif de G_n sur G_{n+1}; soit d_n son degré. Considérant θ_n, pour $n \geqq n_1$, comme un morphisme bijectif de G_n sur G', on voit que θ_{n_1} peut se représenter (si $n > n_1$) comme composé de θ_n et d'un morphisme bijectif $G_{n_1} \rightarrow G_n$ de degré $d_{n_1} \cdots d_{n-1}$ (à savoir le composé de $\omega_{n-1}, \cdots, \omega_{n_1}$).

On en conclut que $d_{n_1} \cdots d_{n-1}$ est au plus égal au degré de θ_{n_1}, donc que $d_n = 1$ pour tout n assez grand, soit pour $n \geqq n_0$. Si $n \geqq n_0$, ω_n est un morphisme bijectif et birationnel de G_n sur G_{n+1}; G_{n+1} étant une variété normale, il en résulte que ω_n est un isomorphisme.

Supposons réciproquement la condition satisfaite. Un raisonnement classique montre alors qu'il existe un groupe algébrique P et un homomorphisme algébrique injectif π de P dans $\mathfrak{G}(U)$ qui possèdent la propriété suivante: pour tout couple (G', γ') formé d'un groupe algébrique G' et d'un homomorphisme algébrique injectif γ' de G' dans $\mathfrak{G}(U)$, tout homomorphisme ω de P dans G' tel que $\pi = \gamma' \circ \omega$ est un isomorphisme. On va montrer que (P, π) est une variété de Picard de U. Soient G un groupe algébrique et γ un homomorphisme algébrique de G dans $\mathfrak{G}(U)$. L'application $(x, s) \to \pi(x) + \gamma(s)$ $(x \in P, s \in G)$ est un homomorphisme algébrique de $P \times G$ dans $\mathfrak{G}(U)$; soit N son noyau; soient G' le groupe $(P \times G)/N$ et ζ l'application canonique de $P \times G$ sur G. On sait que N est un sous-groupe fermé et que l'application γ' de G' dans $\mathfrak{G}(U)$ définie par la condition que $\gamma'(\zeta(x, s)) = \pi(x) + \gamma(s)$ $((x, s) \in P \times G)$ est un homomorphisme algébrique (Proposition 2, I, § V); cet homomorphisme est injectif. Soit e_G l'élément neutre de G; soit ω l'application $x \to \zeta(x, e_G)$ de P dans G'. Il est clair que ω est un homomorphisme et que $\pi = \gamma' \circ \omega$; ω est donc un isomorphisme. Soit e_P l'élément neutre de P; en composant avec ω^{-1} l'homomorphisme $s \to \omega(e_P, s)$ de G dans G', on obtient un homomorphisme φ de G dans P; il est clair que $\pi = \gamma \circ \varphi$. Comme π est injectif, il n'y a qu'un seul homomorphisme φ de G dans P tel que $\pi = \gamma \circ \varphi$; (P, π) est donc bien une variété de Picard de U.

THÉORÈME 1. *Soit U une variété normale semi-complète qui admet une variété de Picard (P, π). Si f est une famille algébrique de classes de diviseurs de U paramétrée par une variété normale T, et si t_0 est un point de T, il existe un morphisme g et un seul de T dans P tel que l'on ait $f(t) = \pi(g(t)) + f(t_0)$ pour tout $t \in T$; de plus, P est une variété complète.*

Nous considérerons d'abord le cas où T est une courbe. Comme T est normale, il est bien connu que T est isomorphe à une sous-variété ouverte d'une courbe normale et complète C; soit J la jacobienne de C. Faisant usage de la Proposition 3, II, § II, on voit que, si χ est une application canonique de C dans J, il y a un homomorphisme algébrique f_1 de J dans $\mathfrak{G}(U)$ tel que l'on ait $f(t) = f_1(\chi(t)) + \mathfrak{c}_0$ pour tout $t \in T$, $\mathfrak{c}_0$ étant un certain point de $\mathfrak{G}(U)$. Il existe un homomorphisme φ de J dans P tell que $f_1 = \pi \circ \varphi$. Posons $g(t) = \varphi(\chi(t)) - \varphi(\chi(t_0))$ $(t \in T)$; il est clair que g est un morphisme de T dans P, et l'on a, si $t \in T$,

$$f(t)-f(t_0)=f_1(\chi(t)-\chi(t_0))=\pi(g(t)).$$

Le morphisme g est caractérisé de manière unique par cette condition puisque π est injectif. Ceci démontre la première assertion dans le cas où T est une courbe. De plus, on notera que $f_1(J)$ est un sous-groupe *abélien* (i. e. complet) de P.

Comme toute suite croissante de sous-groupes connexes complets de P est constante à partir d'un certain rang, il existe un sous-groupe abélien maximal P' de P. Si P'_1 est un sous-groupe abélien quelconque de P, le groupe $P'+P'_1$, image par un homomorphisme du groupe abélien $P'\times P'_1$, est abélien, donc identique à P', ce qui montre que $P'_1\subset P'$. Il résulte de la première partie de la démonstration que, si f est une application algébrique d'une courbe normale T dans $\mathfrak{G}(U)$, on a $f(t)-f(t')\in\pi(P')$ quels que soient t et t' dans T.

Soit maintenant T une variété normale quelconque; soient f une application algébrique de T dans $\mathfrak{G}(U)$ et t_0 un point de T. Montrons que l'on a $f(t)-f(t_0)\in\pi(P')$ pour tout $t\in T$. Soit A l'ensemble des points t tels que $f(t)-f(t_0)\in\pi(P')$; il suffira de montrer que A est dense dans T. En effet, l'ensemble des points $(t,x)\in T\times P'$ tels que $f(t)-f(t_0)=\pi(x)$ est fermé; comme P' est complète, il en résulte que l'ensemble A est fermé. Montrons que toute courbe Γ contenue dans T et passant par t_0 est contenue dans A. Il existe une courbe normale Γ' et un morphisme r de Γ' dans Γ tels que (Γ',r) soit un revêtement de Γ. L'application $t'\to f(r(t'))$ $(t'\in\Gamma')$ est une application algébrique de Γ' dans $\mathfrak{G}(U)$. Par ailleurs, si $t\in\Gamma$, il existe toujours un point $t'\in\Gamma'$ tel que $r(t')=t$; désignant par t_0' un point tel que $r(t_0')=t_0$, on a $f(t)-f(t_0)=f(r(t'))-f(r(t_0'))\in\pi(P')$ en vertu de ce qui a été dit plus haut. Pour montrer que $A=T$, il suffira donc de montrer que la réunion des courbes tracées sur T et passant par t_0 est une partie dense de T. Mais cela résulte du fait que, si E est une partie fermée $\neq T$ de T, il existe au moins une courbe tracée sur T, passant par t_0 et non contenue dans E ([2], chap. III, § III, Proposition 1).

Appliquons ceci au cas où $T=P$, $f=\pi$, t_0 étant l'élément neutre de P; on a $\pi(P)\subset\pi(P')$, d'où $P=P'$ puisque π est injectif. Ceci montre que P est une variété complète.

Revenons à la consideration de la variété T. Soit H l'ensemble des points $(t,x)\in T\times P$ tels que $f(t)-f(t_0)=\pi(x)$. On sait que cet ensemble est fermé. Il résulte de ce qu'on vient de dire que la restriction p à H de la projection $T\times P\to T$ est une application surjective de H sur T. Comme π est injectif, l'application p est en fait bijective. Si H_1 est une composante

irréductible de H telle que $p(H_1)$ soit dense dans T, $p(H_1)$ est égal à T comme il résulte de ce que H_1 est fermé dans $T \times P$ et de ce que P est complète; comme p est bijectif, on a $H_1 = H$; H est donc une sous-variété fermée de $T \times P$ et p est un morphisme bijectif de H sur T. Si nous montrons que p est birationnel, le théorème sera établi; en effet, composant p^{-1} avec la restriction à H de la projection $T \times P \to P$, on obtiendra un morphisme g de T dans P tel que $f(t) - f(t_0) = \pi(g(t))$ pour tout $t \in T$, et g sera uniquement determiné par cette condition puisque π est injectif.

Observons d'abord que, si Δ est une courbe sur H telle que $\Gamma = p(\Delta)$ soit une courbe normale, la restriction p_Δ de p à Δ est un morphisme birationnel de Δ sur Γ. Soit en effet t_1 un point de Γ. Il existe un morphisme g_1 de Γ dans P tel que l'on ait $f(t) - f(t_1) = \pi(g_1(t))$ pour tout $t \in \Gamma$. Comme $f(t_1) - f(t_0) \in \pi(P)$, il en résulte qu'il y a un morphisme g_2 de Γ dans P tel que l'on ait $f(t) - f(t_0) = \pi(g_2(t))$ $(t \in \Gamma)$; Δ n'est alors autre que l'ensemble des points $(t, g_2(t))$ pour $t \in \Gamma$, ce qui montre que p_Δ est un isomorphisme de Δ sur Γ. Nous sommes donc ramenés à établir le lemme suivant:

LEMME 1. *Soit p un morphisme bijectif d'une variété H dans une variété T. Si p est de degré > 1, il existe une courbe Δ sur H telle que $p(\Delta)$ soit normale et que la restriction de p à Δ soit de degré > 1.*

Soit π le cohomomorphisme de p; soient $F(T)$ et $F(H)$ les corps des fonctions numériques sur T et sur H. Désignons par q la caractéristique de K, qui est > 0 en vertu des hypothèses faites. Comme p est radiciel, il y a une fonction $\theta \in F(T)$ qui n'est pas puissance q-ième dans $F(T)$ mais qui est telle que $\pi(\theta)$ soit puissance q-ième dans $F(H)$. On sait qu'il existe alors une dérivation X du corps $F(T)$ telle que $X(\theta) \neq 0$. A cette dérivation est associé un champ de vecteurs tangents à T défini sur une partie ouverte non vide T_1 de T, champ de vecteurs que nous désignerons encore par X. Soient par ailleurs H_0 et T_0 les ensembles de points simples de H et T; $p(H_0)$ contient une partie ouverte non vide de T. Il en résulte qu'il y a un point $t_0 \in p(H_0) \cap T_0 \cap T_1$ tel que θ soit définie en t_0 et que $(X(\theta))(t_0) \neq 0$, ce qui signifie que $< X(t_0), \theta > \neq 0$. Comme t_0 est un point simple de T, il y a une courbe Γ_1 de T, passant par t_0, y admettant un point simple, telle que $X(t_0)$ soit tangent à Γ_1 en t_0. Si θ_1 est l'empreinte de θ sur cette courbe, on a $< X(t_0), \theta_1 > \neq 0$, ce qui montre que θ_1 n'est pas puissance p-ième dans le corps des fonctions numériques sur Γ_1. Comme T est normale en t_0 et comme $p^{-1}(t_0)$ se compose d'un seul point t_0', il y a une courbe Δ_1 passant par t_0' telle que $p(\Delta_1)$ soit dense dans Γ_1 ([2], chap. V, § V, Proposition 2). Soit p_1 la restriction de p à Δ_1. Montrons que $\theta_1 \circ p_1$ est puissance q-ième dans

le corps des fonction numériques sur Δ_1. La fonction $\theta \odot p$ est puissance q-ième d'une fonction θ' sur H; de plus, elle est définie au point t_0' en lequel H est normale puisque $t_0' \in H_0$; θ' est donc définie en t_0' et admet une empreinte θ'_1 sur Δ_1; il est alors clair que $\theta_1 \odot p_1 = \theta'^q_1$. Ceci montre que p_1 est de degré > 1. L'ensemble $p_1(\Delta_1)$ contient une sous-variété ouverte normale Γ de Γ_1; il suffit alors de prendre $\Delta = p^{-1}(\Gamma)$.

Remarque. Soit U une variété normale et complète sur laquelle on fait l'hypothèse suivante: il existe un groupe algébrique *complet* P et un homomorphisme algébrique injectif π de P dans $\mathfrak{G}(U)$ tels que, pour tout groupe algébrique *complet* G et tout homomorphisme algébrique γ de G dans $\mathfrak{G}(U)$, il existe un homomorphisme φ et un seul de G dans P tel que $\gamma = \pi \circ \varphi$. Alors (P, π) est variété de Picard de U. Reprenons en effet la démonstration précédente. La jacobienne d'une courbe étant une variété complète, la première partie de la démonstration établit encore que, si T est une courbe normale et f une application algébrique de T dans $\mathfrak{G}(U)$, et si $t_0 \in T$, il existe un morphisme g et un seul de T dans P tel que $f(t) - f(t_0) = \pi(g(t))$ pour tout $t \in T$. Le reste de la démonstration se transporte alors sans modification, et établit que, si T est une variété normale quelconque et f une application algébrique de T dans $\mathfrak{G}(U)$, et si $t_0 \in T$, il existe un morphisme g et un seul de T dans P tel que $f(t) - f(t_0) = \pi(g(t))$ $(t \in T)$. Appliquons ceci au cas où T est un groupe algébrique et g un homomorphisme algébrique, t_0 étant l'élément neutre du groupe, d'où $f(t_0) = 0$; comme π est un homomorphisme injectif, il en résultera que g est un homomorphisme; (P, π) est donc bien une variété de Picard de U. Ceci étant, on voit que, si U est normale et semi-complète, il suffit pour que U admette une variété de Picard, que la condition énoncée dans la Proposition 2 soit satisfaite quand on y suppose de plus que les G_n sont des groupes complets; il suffit en effet de reprendre la démonstration de la Proposition 2 en n'y considérant que des groupes complets: on montre ainsi qu'il existe un groupe complet P et un homomorphisme algébrique injectif π de P dans $\mathfrak{G}(U)$ qui possèdent la propriété énoncée au début de cette remarque.

PROPOSITION 3. *Soient U et U' des variétés normales et complètes. Supposons que U' admette une variété de Picard (P', π') et qu'il existe un morphisme surjectif f de U' dans U. Alors U admet une variété de Picard.*

Soient G un groupe algébrique complet et γ un homomorphisme algébrique injectif de G dans $\mathfrak{G}(U)$. Alors l'application γ': $s \to f^*(\gamma(s))$ est un homomorphisme algébrique de G dans $\mathfrak{G}(U')$. Nous allons montrer qu'il est de noyau fini. Soit N la composante algébrique de l'élément neutre dans son

noyau; N est une variété complète. Il en résulte que l'homomorphisme $f^*: \mathfrak{N}(U,N) \to \mathfrak{N}(U',N)$ est injectif (Proposition 4, I, § IV). En vertu des isomorphismes canoniques $\mathfrak{N}(U,N) \cong \mathfrak{N}(N,U)$, $\mathfrak{N}(U',N) \cong \mathfrak{N}(N,U')$, l'homomorphismes $f^*: \mathfrak{N}(N,U) \to \mathfrak{N}(N,U')$ est injectif. Soit γ_0 la restriction de γ à N; cette application de N dans $\mathfrak{G}(U)$ est définie par un élément $m \in \mathfrak{M}(N,U)$ dont l'image par l'homomorphisme $f: \mathfrak{M}(N,U) \to \mathfrak{M}(N,U')$ est nulle. Il résulte alors de ce que nous venons de dire que l'image de m dans $\mathfrak{N}(N,U)$ est nulle, c'est-à-dire que γ_0 est constante. Comme c'est un homomorphisme, c'est l'application nulle; comme γ est injectif, N se réduit à son élément neutre. Ceci étant, soit (G_n, γ_n) une suite de couples formés chacun d'un groupe algébrique complet G_n et d'un homomorphisme algébrique injectif γ_n de G_n dans $\mathfrak{G}(U)$; soit de plus, pour tout n, ω_n un homomorphisme de G_n dans G_{n+1} tel que $\gamma_n = \gamma_{n+1} \circ \omega_n$. Soit γ_n' l'application $s \to f^*(\gamma_n(s))$ de G_n dans $\mathfrak{G}(U')$; il existe donc un homomorphisme algébrique θ_n et un seul de G_n dans P' tel que $\gamma_n' = \pi' \circ \theta_n$. Il est clair que l'on a $\theta_n = \theta_{n+1} \circ \omega_n$. Soit N_n le noyau de θ_n; on a donc $\omega_n(N_n) \subset N_{n+1}$, et ω_n définit par passage aux quotients un homomorphisme ω_n' du groupe $G_n' = G_n/N_n$ dans G_{n+1}'. Par ailleurs θ_n définit par passage aux quotients un homomorphisme injectif θ_n' de G_n' dans P', et on a $\theta_n' = \theta_{n+1}' \circ \omega_n'$. Appliquant alors la Proposition 2 aux homomorphismes algébriques $\pi' \circ \theta_n'$ des G_n' dans $\mathfrak{G}(U')$, on voit qu'il existe un n_1 tel que ω_n' soit un isomorphisme pour tout $n \geqq n_1$. Si donc $n \geqq n_1$, on a $\dim G_{n+1} = \dim G_{n+1}' = \dim G_n' = \dim G_n$, et, comme ω_n est évidemment injectif (les γ_n l'étant), il en résulte que ω_n est aussi surjectif. L'homomorphisme obtenu en composant ω_n avec l'application canonique $G_{n+1} \to G_{n+1}'$ s'obtient aussi en composante l'application canonique $G_n \to G_n'$ avec l'isomorphisme ω_n'; il est donc séparable, d'où il résulte que ω_n est lui-même séparable. Etant bijectif, il est birationnel; comme G_{n+1} est normale, ω_n est un isomorphisme. Tenant compte de la remarque qui suit la démonstration du Théorème 1, on voit que U admet une variété de Picard.

II. Variete de Picard d'un produit. Soit U une variété normale et semi-complète qui admet une variété de Picard (P, π). Soit T une variété normale; nous identifierons les applications algébriques de T dans $\mathfrak{G}(U)$ aux éléments du groupe $\mathfrak{M}(T,U)$. Si t_0 est un point de T, nous désigerons par $\mathfrak{M}(t_0; T, U)$ le groupe des applications algébriques de T dans $\mathfrak{G}(U)$ qui appliquent t_0 sur 0. Ce groupe est canoniquement isomorphe à $\mathfrak{N}(T,U)$. En effet, le noyau de l'application canonique $\mathfrak{M}(T,U) \to \mathfrak{N}(T,U)$ se compose des applications constantes de T dans $\mathfrak{G}(U)$ et n'a par suite que 0 en commun avec $\mathfrak{M}(t_0; T, U)$. Par ailleurs si $f \in \mathfrak{M}(T,U)$, la formule $f(t) = (f(t) - f(t_0)) + f(t_0)$ montre

que l'image de f dans $\mathfrak{N}(T, U)$ est aussi l'image d'un élément de $\mathfrak{M}(t_0; T, U)$. Il résulte du Théorème 1, § I que l'application $g \to \pi \circ g$ est un isomorphisme du groupe $\mathrm{Mor}(t_0; T, P)$ des morphismes de T dans P qui appliquenet t_0 sur 0 sur le groupe $\mathfrak{M}(t_0; T, U)$. Les trois groupes

$$\mathfrak{N}(T, U), \qquad \mathfrak{M}(t_0; T, U); \qquad \mathrm{Mor}(t_0; T, P)$$

sont donc canoniquement isomorphes les uns aux autres, Par ailleurs, si on se donne des morphismes f d'une variété normale T' dans T et g d'une variété semi-complète et normale U' dans U, ainsi qu'un point $t_0' \in T'$ tel que $f(t_0') = t_0$, f et g définissent un homomorphisme $r_1 : \mathfrak{N}(T, U) \to \mathfrak{N}(T', U')$; de plus, si $\varphi \in \mathfrak{M}(t_0; T, U)$, l'application $t' \to g^*(\varphi(f(t')))$ appartient à $\mathfrak{M}(t_0'; T', U')$, ce qui définit un homomorphisme $r_2 : \mathfrak{M}(t_0; T, U) \to \mathfrak{M}(t_0'; T', U')$. Enfin, si U' admet une variété de Picard (P', π'), g définit un homomorphisme $\bar{g}$: $P \to P'$, et l'application $\varphi \to \bar{g} \circ \varphi \circ f$ est un homomorphisme r_3 de $\mathrm{Mor}(t_0; T, P)$ dans $\mathrm{Mor}(t_0'; T', P')$. On vérifie immédiatement que le diagramme

$$\begin{array}{ccccc}
\mathfrak{N}(T, U) & \longrightarrow & \mathfrak{M}(t_0; T, U) & \longrightarrow & \mathrm{Mor}(t_0; T, P) \\
{\scriptstyle r_1}\downarrow & & {\scriptstyle r_2}\downarrow & & {\scriptstyle r_3}\downarrow \\
\mathfrak{N}(T', U') & \longrightarrow & \mathfrak{M}(t_0'; T', U') & \longrightarrow & \mathrm{Mor}(t_0'; T', P'),
\end{array}$$

où les lignes horizontales sont données par les homomorphismes canoniques mentionnés ci-dessus, est commutatif.

Proposition 1. *Soient U, V, W des variétés normales dont l'une au moins est semi-complète et admet une variété de Picard; soient p, q, r les projections de $U \times V \times W$ sur $U \times V$, $V \times W$ et $W \times U$ respectivement. Le groupe $\mathfrak{G}(U \times V \times W)$ est alors la somme des images par p^*, q^*, r^* des groupes $\mathfrak{G}(U \times V)$, $\mathfrak{G}(V \times W)$ et $\mathfrak{G}(W \times U)$.*

Supposons que W soit semi-complète et admette une variété de Picard (P, π). Si r_W est la projection $U \times V \times W \to W$, $\mathfrak{N}(U \times V, W)$ est

$$\mathfrak{G}(U \times V \times W) / (p^*(\mathfrak{G}(U \times V)) + r_W^*(\mathfrak{G}(W))).$$

Choisissons par ailleurs un point $x_0 \in U$ et un point $y_0 \in V$; $\mathfrak{N}(U \times V, W)$ est alors isomorphe à $\mathrm{Mor}((x_0, y_0); U \times V, P)$. Or, P étant une variété abélienne, si φ est une fonction sur $U \times V$ à valeurs dans P, il existe des fonctions φ_U sur U et φ_V sur V à valeurs dans P telles que l'on ait $\varphi(x, y) = \varphi_U(x) + \varphi_V(y)$ pour tout point simple $(x, y) \in U \times V$. Si φ est un morphisme, φ_U et φ_V sont des morphismes; en effet, y_1 étant un point simple de V, φ_U coincide évidemment avec le morphisme $x \to \varphi(x, y_1) - \varphi_V(y_1)$, et on voit de même que φ_V est un morphisme. De plus, si on suppose que

$\varphi(x_0, y_0) = 0$, on peut supposer que $\varphi_U(x_0) = \varphi_V(y_0) = 0$, et φ_U et φ_V sont alors uniquement determinés. Soient s_U et s_V les projections de $U \times V$ sur U et sur V; il résulte de ce qu'on vient de dire que l'application $(\varphi_U, \varphi_V) \to \varphi_U \circ s_U + \varphi_V \circ s_V$ est un isomorphisme du groupe

$$\mathrm{Mor}(x_0; U, P) \times \mathrm{Mor}(y_0; V, P) \text{ sur } \mathrm{Mor}((x_0, y_0); U \times V, P).$$

On en conclut que $\mathfrak{N}(U \times V, W)$ est somme directe des groups $s_U^*(\mathfrak{N}(U, W))$ et $s_V^*(\mathfrak{N}(V, W))$, et par suite que $\mathfrak{G}(U \times V \times W)$ est somme des groupes $p^*(\mathfrak{G}(U \times V))$, $q^*(\mathfrak{G}(V \times W))$, $r^*(\mathfrak{G}(W \times U))$ et $r_W^*(\mathfrak{G}(W))$; le dernier de ces groupes étant contenu dans le deuxième (et dans le troisième), la Proposition 1 est établie.

THÉORÈME 2. *Soient U_1 et U_2 des variétés normales et semi-complètes qui admettent des variétés de Picard (P_1, π_1) et (P_2, π_2). Soit q_i la projection de $U_1 \times U_2$ sur U_i $(i = 1, 2)$; soit π l'homomorphisme algébrique*

$$(x_1, x_2) \to q_1^*(\pi_1(x_1)) + q_2^*(\pi_2(x_2))$$

de $P_1 \times P_2$ dans $\mathfrak{G}(U_1 \times U_2)$. Alors $(P_1 \times P_2, \pi)$ est une variété de Picard de $U_1 \times U_2$.

Soit G un groupe algébrique. Il résulte immédiatement de la proposition 1 que $\mathfrak{N}(G, U_1 \times U_2)$ est somme des groupes $q_i^*(\mathfrak{N}(G, U_i))$ $(i = 1, 2)$. Soit γ un homomorphisme algébrique de G dans $\mathfrak{G}(U_1 \times U_2)$; il résulte de ce qu'on vient de dire qu'il existe des applications algébriques γ_1 et γ_2 de G dans $\mathfrak{G}(U_1)$ et $\mathfrak{G}(U_2)$ et un élément $\mathfrak{c} \in \mathfrak{G}(U_1 \times U_2)$ tels que

$$\gamma(s) = q_1^*(\gamma_1(s)) + q_2^*(\gamma_2(s)) + \mathfrak{c}$$

pour tout $s \in G$. Soit e l'élément neutre de G; on peut évidemment supposer que $\gamma_1(e) = \gamma_2(e) = 0$, et on a alors $\mathfrak{c} = 0$. Soit x_i un point de U_i $(i = 1, 2)$; soit j_1 (resp. j_2) l'application $y_1 \to (y_1, x_2)$ (resp. $y_2 \to (x_1, y_2)$) de U_1 (resp. U_2) dans $U_1 \times U_2$. Alors $q_2 \circ j_1$ est l'application constante de valeur x_2 de U_1 dans U_2. Il en résulte que

$$j_1^*(q_1^*(\gamma_1(s))) = \gamma_1(s), \qquad j_1^*(q_2^*(\gamma_2(s))) = 0,$$

et par suite que $\gamma_1(s) = j_1^*(\gamma(s))$, ce qui montre que γ_1 est un homomorphisme; on verrait de même que γ_2 est un homomorphisme. Il existe donc un homomorphisme g_i: $G \to P_i$ tel que $\gamma_i = \pi_i \circ g_i$ $(i = 1, 2)$. L'application g: $s \to (g_1(s), g_2(s))$ est un homomorphisme de G dans $P_1 \times P_2$, et il est clair que $\gamma = \pi \circ g$. Pour montrer que g est uniquement determiné par cette condition, il suffit d'établir que π est injectif; or cela résulte immédiatement des formules $j_i^*(\pi(x_1, x_2)) = \pi_i(x_i)$ $(i = 1, 2)$. Le théorème 2 est donc établi.

Soient C une courbe normale et complète et J sa jacobienne. Désignons par ι l'application identique de J dans $\mathfrak{G}(C)$; alors (J,ι) est une variété de Picard de C. En effet, il est clair que ι est un homomorphisme algébrique de J dans $\mathfrak{G}(C)$. Soient maintenant G un groupe algébrique, e son élément neutre et γ un homomorphisme algébrique de G dans $\mathfrak{G}(C)$. Il résulte du théorème 1, § I que γ est un morphisme de G dans J; comme ι est injectif, γ est un homomorphisme de G dans J et est le seul homomorphisme g de G dans J tel que $\iota \circ g = \gamma$, ce qui établit notre assertion.

Il résulte alors du théorème 2 que tout produit de courbes normales et complètes admet une variété de Picard. Soit maintenant U une variété abélienne; elle peut être munie d'une structure de groupe commutatif, dont nous désignerons l'élément neutre par x_0. Montrons que, si $r = \dim U$, il existe des courbes complètes $\Gamma_1, \cdots, \Gamma_r$ tracées sur U telles qu'il existe un morphisme surjectif de $\Gamma_1 \times \cdots \times \Gamma_r$ sur U. Supposons déjà construites des courbes complètes Γ_i pour $i < k$, tracées sur U et passant par x_0, telles que l'application $(x_i)_{i<k} \to \sum_{i<k} x_i$ applique $\prod_{i<k} \Gamma_i$ sur une sous-variété U_{k-1} de dimension $k-1$ de U (k étant un entier entre 1 et r). Comme $k-1 < r$, il y a une courbe Γ_k sur U, passant par x_0 mais non contenue dans U_{k-1}; on peut supposer Γ_k fermée dans U, et Γ_k est alors complète. L'image de $\prod_{i \leqq k} \Gamma_i$ par l'application $(x_i)_{i\leqq k} \to \sum_{i \leqq k} x_i$ est une sous-variété U_k de U (puisque les Γ_i sont complètes) qui contient U_{k-1} et Γ_k (puisque les Γ_i passent par x_0) et qui est par suite de dimension $\geqq k$; comme $\prod_{i \leqq k} \Gamma_i$ est de dimension k, U_k est de dimension $\leqq k$. Ceci étant, on a $U_r = U$, ce qui démontre notre assertion. Pour tout $i \leqq r$, il existe un morphisme surjectif d'une courbe normale et complète C_i sur Γ_i; il existe donc un morphisme surjectif de la variété $C_1 \times \cdots \times C_r$ sur U; tenant compte de la proposition 3, § I, on en déduit la

PROPOSITION 2. *Toute variété abélienne admet une variété de Picard.*

III. Variete d'Albanese stricte. Soit U une variété. Il est bien connu qu'il existe une variété abélienne A et une fonction dominante f sur U à valeurs dans A qui possèdent la propriété suivante: si g est une fonction quelconque sur U à valeurs dans une variété abélienne B, il existe un morphisme h et un seul de A dans B tel que $g = h \odot f$; de plus, si (A', f') est un autre couple qui possède les mêmes propriétés que (A, f), il existe un isomorphisme j et un seul de A sur A' tel que $f' = j \odot f$; on dit que (A, f) est une *variété d'Albanese* de U. Nous allons voir maintenant qu'il existe un énoncé analogue relatif au cas où on considère des morphismes de U dans

des variétés abéliennes au lieu de fonctions. Soit (A, f) une variété d'Albanese de A; A peut être munie d'une structure de groupe commutatif, dont nous dsignerons l'élément neutre par e. Soit C la correspondance entre U et A associée à f; c'est l'adhérence dans $U \times A$ du graphe de f. Pour tout $x \in U$, soit $E(x)$ l'ensemble des points $a \in A$ tels que $(a, x) \in C$; si f est définie en x (en particulier, si x est simple sur U), $E(x)$ se compose du seul point $f(x)$. Nous désignerons par $E'(x)$ l'ensemble des points de la forme $a' - a$, avec a et a' dans $E(x)$; enfin, nous désignerons par N le plus petit sous-groupe fermé de A contenant les ensembles $E'(x)$ pour tous les $x \in U$. Soit A' la variété abélienne A/N, et soit ω l'application canonique de A sur A'. Posons $f' = \omega \odot f$; montrons que f' est un morphisme de U dans A'. Soit C' la correspondance entre A' et U associée à f'. L'application $\omega_1 : (a, x) \to (\omega(a), x)$ de $A \times U$ dans $A' \times U$ est propre puisque ω est propre (en tant que morphisme d'une variété complète); $\omega_1(C)$ est donc une sous-variété fermée de $A' \times U$. Cette variété est l'adhérence de l'ensemble des points $(f'(x), x)$, x parcourant l'ensemble des points en lesquels f est définie; il en résulte immédiatement que c'est aussi l'adhérence du graphe de f', donc que $C' = \omega_1(C)$. Soit x un point quelconque de U; comme A' est complète, il y a au moins un point $a' \in A'$ tel que $(a', x) \in C'$. Montrons qu'il n'y en a qu'un. Soient a_1' et a_2' des points de A' tels que $(a_i', x) \in C'$ $(i = 1, 2)$. Comme $C' = \omega_1(C)$, il y a des points a_i $(i = 1, 2)$ de A tels que $(a_i, x) \in C$, $a_i' = \omega(a_i)$; mais on a alors $a_2 - a_1 \in E'(x) \subset N$, d'où $a_1' = a_2'$, ce qui établit notre assertion. Ceci étant, il résulte du fait que A' est normale et du théorème principal de Zariski que f' est partout définie, donc que c'est un morphisme. Soit maintenant g un morphisme de U dans une variété abélienne B; soit h le morphisme de A dans B tel que $g = h \odot f$. Soient x un point de U et a_1, a_2 des points de $E(x)$; comme h est définie en ces points, les points $(h(a_1), x)$ et $(h(a_2), x)$ appartiennent à la correspondance entre B et U associée à $h \odot f$ ([2], chap. IV, § I, Proposition 5). Comme $h \odot f = g$ est un morphisme, il en résulte que $h(a_1) = h(a_2) = g(x)$. Or B peut être munie d'une structure de groupe commutatif admettant $h(e)$ comme élément neutre; il est alors bien connu que h est un homomorphisme de A dans B, d'où $h(a_1 - a_2) = 0$. Le noyau de h contient donc tous les ensembles $E'(x)$, ce qui montre qu'il contient N. On en déduit que h se met sous la forme $h' \circ \omega$, où h' est un morphisme de A' dans B; il est clair que l'on a $g = h' \circ f'$. Comme il n'existe qu'un seul morphisme h de A dans B tel que $g = h \circ f$, il n'existe qu'un seul morphisme h' de A' dans B tel que $g = h' \circ f'$. Nous avons donc établi le

THÉORÈME 3. *Soit U une variété. Il existe une variété abélienne A et un morphisme f de U dans A qui possèdent la propriété suivante: si g est*

un morphisme de U dans une variété abélienne B, il existe un morphisme h et un seul de A dans B tel que $g = h \circ f$.

Let notations étant celles du théorème précédent, nous dirons que (A, f) est une *variété d'Albanese stricte de la variété U.*

On notera que, si (A_0, f_0) est une variété d'Albanese de U et (A, f) une variété d'Albanese stricte, A est une variété quotient de A_0; par ailleurs, si U est non singulière, on a $A = A_0$. Le nombre $\nu(U) = \dim A_0 - \dim A$ est donce un indicateur de l'importance des singularités de U. Il n'est pas sans interêt à ce sujet d'observer qu'il existe toujours une variété U' telle que $\nu(U') = 0$ qui admet un morphisme surjectif birationnel sur U: il suffit en effet de prendre pour U' la correspondance entre A_0 et U associée à la fonction f_0. Il y aurait peut être lieu d'examiner si l'opération qui consiste à passer de U à une variété telle que U' ne serait pas utile dans l'étude du problème de la réduction des singularités de U.

IV. Variete de Picard d'une variete normale complete.

THÉORÈME 4. *Toute variété normale semi-complète U admet une variété de Picard.*

Soit (A, f) une variété d'Albanese stricte de U. Comme A est une variété abélienne, elle admet une variété de Picard (P, α). L'application $\pi: z \to f^*(\alpha(z))$ $(z \in P)$ est un homomorphisme algébrique de P dans $\mathfrak{G}(U)$. Nous allons montrer que (P, π) est une variété de Picard de U.

Soit G un groupe algébrique complet. Le morphisme $f: U \to A$ définit un homomorphisme $f^*: \mathfrak{R}(G, A) \to \mathfrak{R}(G, U)$; montrons que set homomorphisme est un isomorphisme. Il suffit de montrer que l'homomorphisme

$$f^*: \mathfrak{R}(A, G) \to \mathfrak{R}(U, G)$$

est un isomorphisme. Nous désignerons par x_0 un point de U; la variété A possède une structure de groupe commutatif admettant $f(x_0) = e$ comme élément neutre. Par ailleurs, G, qui est une variété abélienne, admet une variété de Picard (Q, χ). Il résulte alors de ce qui a été dite au début du § II qu'il existe des isomorphismes canoniques $\mathfrak{R}(A, G) \to \mathrm{Mor}(e; A, Q)$, $\mathfrak{R}(U, G) \to \mathrm{Mor}(x_0; U, Q)$ tels que le diagramme

$$\begin{tikzcd} \mathfrak{R}(A, G) \arrow[r, "f^*"] \arrow[d] & \mathfrak{R}(U, G) \arrow[d] \\ \mathrm{Mor}(e; A, Q) \arrow[r] & \mathrm{Mor}(x_0; U, Q) \end{tikzcd}$$

soit commutatif, la deuxième flèche horizontale de ce diagramme étant l'application $h \to h \circ f$ ($h \in \mathrm{Mor}(e; A, Q)$). Or il résulte de la définition des variétés d'Albanese strictes que l'application $h \to h \circ f$ est un isomorphisme de $\mathrm{Mor}(e; A, Q)$ sur $\mathrm{Mor}(x_0; U, Q)$. Il en résulte bien que f^* est un isomorphisme.

Puisque (P, α) est une variété de Picard de A, il y a un isomorphisme canonique de $\mathfrak{N}(G, A)$ sur $\mathrm{Mor}(e; G, P)$, d'où, en tenant compte de l'isomorphisme f^*, un isomorphisme

$$\mathrm{Mor}(e; G, P) \cong \mathfrak{N}(G, U).$$

Cet isomorphisme s'explicite comme suit; $\mathfrak{N}(G, U)$ étant identifié à un groupe quotient de $\mathfrak{M}(G, U)$, donc à un quotient du groupe des applications algébriques de G dans $\mathfrak{G}(U)$, l'élément de $\mathfrak{N}(G, U)$ qui correspond à un élément g de $\mathrm{Mor}(e; G, P)$ est la classe dans $\mathfrak{N}(G, U)$ de l'application $s \to f^*(\alpha(g(s)))$ de G dans $\mathfrak{G}(U)$. Ceci dit, soit γ un homomorphisme algébrique de G dans $\mathfrak{G}(U)$; la classe de γ dans $\mathfrak{N}(G, U)$ correspond par l'isomorphisme précédent à un élément $g \in \mathrm{Mor}(e; G, P)$; comme g est un morphisme de la variété abélienne G dans P et comme $g(e) = 0$, g est un homomorphisme de G dans P. Les applications $s \to f^*(\alpha(g(s))) = \pi(g(s))$ et γ, qui ont même image dans $\mathfrak{N}(G, U)$, ne diffèrent que par une application constante de G dans $\mathfrak{G}(U)$; comme elles appliquent toutes deux e sur 0, elles sont égales. Il ne reste plus qu'à montrer qu'il n'y a qu'un homomorphisme g de G dans P tel que $\gamma = \pi \circ g$, i. e. que si un homomorphisme g de G dans P est tel que $\pi \circ g = 0$, on a $g = 0$. Or g est alors un élément de $\mathrm{Mor}(e; G, P)$ dont l'image dans $\mathfrak{N}(G, U)$ par l'isomorphisme considéré plus haut est nulle; on a donc bien $g = 0$, et le théorème 4 est établi.

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CHARACTERISTIC CLASSES AND HOMOGENEOUS SPACES, III.*

By A. Borel and F. Hirzebruch.

This paper consists of three parts, related to each other only by the fact that they bring complements to [1].

In [1, §§ 25, 26], certain expressions ($\hat{A}$-genus, Chern characters of bundles over spheres, etc.) were proved to be integers "exc 2," that is, up to a power of two. This restriction came from the fact that the proofs relied heavily on the integrality "exc 2" of the Todd genus of an almost complex manifold proved in [5]. Since then Milnor [8, 12] has shown the Todd genus to be an integer. This fact will be used in § 3 to free our earlier results from the powers of two. For this, it will be necessary to generalize slightly the notion of almost complex manifold, and to introduce between vector bundles an equivalence relation (called here S-equivalence), in which the trivial bundles form one class. These preliminaries are dealt with in §§ 1, 2.

In [1, § 23.3], it was proved that the A-genus of a coset space G/U is zero when G and U are compact, connected, semi-simple, of the same rank. The proof made use of a lemma (23.4) stating that the sum of the positive roots of U is singular in G, which was proved essentially by case by case checking. § 4 brings an a priori proof of this lemma, in the framework of the theory of roots. When all roots of G have the same length, 23.4 is equivalent to a theorem of de Siebenthal [10] saying that the "main diagonal" of U is singular in G. We also give a general proof of this result, which is obtained in [10] by case by case checking.

Finally, § 5 gives two elementary sufficient conditions under which the Stiefel-Whitney class $w(M)$ or the Pontrjagin class $\tilde{p}(M)$ (see [1, § 9.3]) of a compact manifold M reduces to 1, which are then applied to G/T.

The notation of [1] will be used freely.

1. S-classes of vector bundles.

1.1. *Notation.* L stands for the field either of real numbers $\boldsymbol{R}$, or of complex numbers $\mathfrak{C}$, or of quaternions $\boldsymbol{K}$. $\boldsymbol{GL}(n,L)$ (resp. $\boldsymbol{U}(n,L)$) is the

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general linear group (resp. unitary group) in L^n. A bundle with typical fibre L^n and structural group $\boldsymbol{GL}(n, L)$ or $\boldsymbol{U}(n, L)$ is called an L-vector bundle.

1.2. Definition. *Let X be a topological space. Two L-vector bundles ξ, η over X are said to be S-equivalent (suspension equivalent) if there exist trivial bundles α, β such that the Whitney sums $\xi \oplus \alpha$ and $\eta \oplus \beta$ are equivalent bundles in the usual sense. The S-equivalence class, or S-class, of ξ will be denoted by $[\xi]$, and $K'(X, L)$ will be the set of S-classes of L-vector bundles over X.*

Let ξ, η be two L-vector bundles. $[\xi] = [\eta]$ means that the associated principal bundles become equivalent after the standard extension of the structural group to $\boldsymbol{U}(N, L)$, for some N, or also that the associated unit sphere bundles become equivalent after iterated suspension of the fibres.

The Whitney sum is commutative, associative, and clearly compatible with S-equivalence. Therefore it defines in $K'(X, L)$ a commutative, associative, operation for which the S-class of the trivial bundle is a zero element.

Let $f\colon X \to Y$ be a continuous map. Then, if we associate to an L-vector bundle over Y the induced bundle on X, we define clearly a homomorphism of $K'(Y, L)$ into $K'(X, L)$.

1.3. Proposition. *Let X be a locally compact, paracompact, finite dimensional space. Then $K'(X, L)$ is a commutative group (with respect to Whitney sum). The S-class of the trivial bundle is the zero element.*

There remains only to show the existence of the inverse. On the Grassmann manifold $\boldsymbol{U}(n+N, L)/\boldsymbol{U}(n, L) \times \boldsymbol{U}(N, L)$ there are two canonical L-vector bundles ξ, η with typical fibres L^n, L^N, whose sum is the trivial bundle. Hence $[\xi] + [\eta] = 0$. Since, by the classification theorem, any L-vector bundle with typical fibre L^n over X is induced from ξ by a map of X into the Grassmannian (for N suitably large), our assertion follows immediately.

1.4. The total Chern class of a complex vector bundle depends only on its S-class, as follows from the multiplication theorem [1, § 9.7]. It belongs to the set $\Gamma(X, \boldsymbol{Z})$ of elements of $H^*(X, \boldsymbol{Z})$ having a zero dimensional term equal to 1, and vanishing odd dimensional components. $\Gamma(X, \boldsymbol{Z})$ is a commutative group under the cup-product, and it is clear that assigning to each complex vector bundle its Chern class yields a group homomorphism of $K'(X, \boldsymbol{C})$ into $\Gamma(X, \boldsymbol{Z})$. An analogous remark can be made of course for the Pontrjagin, symplectic Pontrjagin and Stiefel-Whitney classes.

2. Weakly almost complex structures.

2.1. The standard inclusion of $\boldsymbol{GL}(n,\boldsymbol{C})$ into $\boldsymbol{GL}(2n,\boldsymbol{R})$ induces obviously a homomorphism of $K'(X,\boldsymbol{C})$ into $K'(X,\boldsymbol{R})$ to be denoted by λ. An S-class of real vector bundles is said to admit (to have) a complex structure if it belongs to the image of λ (and if an element of its inverse image has been chosen). A real vector bundle ξ admits (has) a *weak complex structure* if $[\xi]$ admits (has) a complex structure. Thus, a weak complex structure of a real vector bundle ξ is given by a trivial bundle α and a complex structure of $\xi \oplus \alpha$ in the usual sense [1, § 7.3]. Finally, a manifold is *weakly almost complex* (admits a weak almost complex structure) if its tangent bundle has been endowed with (admits) a weak complex structure.

An orientation of a real vector bundle ξ with fibre $\boldsymbol{R}^q$ is a section of the associated bundle with $\boldsymbol{O}(q)/\boldsymbol{SO}(q)$ as fibre. Since

$$\boldsymbol{O}(q)/\boldsymbol{SO}(q) \rightarrow \boldsymbol{O}(q+1)/\boldsymbol{SO}(q+1)$$

is bijective, an orientation of ξ depends only on the S-class of ξ. Thus, a weak complex structure of ξ defines an orientation of ξ. In particular, *a weakly almost complex manifold is canonically oriented.*

2.2. *Chern classes.* The Chern class of a weakly almost complex manifold X is by definition the Chern class of the weak complex structure of its tangent bundle. If X is compact, of dimension $2n$, then $c_n[X]$ is not necessarily the Euler number. For instance, take for X the unit sphere in $\boldsymbol{R}^{2n+1}$. The normal bundle and its Whitney sum with the tangent bundle are trivial. Therefore $\boldsymbol{S}_{2n}$ admits a weak almost complex structure defined by a trivial complex bundle. Then $c_n[\boldsymbol{S}_{2n}] = 0$.

2.3. *Submanifolds of codimension* 2. Let X be a compact weakly almost complex manifold, ξ its real tangent bundle, and $[\xi']$ the complex structure of $[\xi]$. Let $d \in H^2(X,\boldsymbol{Z})$. According to Thom [11] there exists an oriented submanifold D of X, of codimension 2, whose normal bundle ν is a real vector bundle with structure group $\boldsymbol{SO}(2)$ and characteristic class $i^*(d)$, where i is the embedding of D in X. Since $\boldsymbol{SO}(2) = \boldsymbol{U}(1)$, the bundle ν has a complex structure ν', whose total Chern class is $1 + i^*(d)$. Let δ be the real tangent bundle to D. Then we have in $K'(D,\boldsymbol{R})$ the equalities

$$[\delta] = [i^*\xi] - [\nu] = \lambda(i^*[\xi'] - [\nu']),$$

which show that $[\delta]$ admits a complex structure represented by a bundle δ' whose Chern class is

$$c(\delta') = i^*(c(\xi')) \cdot (1 + i^*(d))^{-1}.$$

This proves the following proposition.

2.4. PROPOSITION. *Let X be a compact weakly almost complex manifold and $c(X)$ be its total Chern class. Then every element $d \in H^2(X, \mathbf{Z})$ is representable by a submanifold D of codimension 2 which carries a weakly almost complex structure, whose total Chern class $c(D)$ is equal to $i^*(c(X) \cdot (1+d)^{-1})$, where i is the embedding of D in X.*

2.5. Let X be a compact weakly almost complex manifold of even dimension, $d \in H^2(X, \mathbf{R})$, and η a complex vector bundle over X. The Todd genus $T(X)$, the virtual Todd genus $T(d)_X$ of d, and the number $T(X, \eta)$ are then defined in exactly the same way as for an almost complex manifold. If η is a complex line bundle, with first Chern class a, then $T(X, \eta) = T(X, a)$, where $T(X, a) = T(X) - T(-a)_X$. For all this, see [5, §§ 10-12]; the definitions given there were also recalled in [1, §§ 22.1, 25.1]. It follows then from 2.4 that for every element $d \in H^2(X, \mathbf{Z})$, *the virtual Todd genus $T(d)_X$ is equal to the Todd genus of some compact weakly almost complex manifold.*

2.6. Milnor [8] (see also [12]) has established a complex analogue of cobordism theory, and has proved that the *Todd genus of a weakly almost complex manifold is an integer.* This (and 2.5) yield the

PROPOSITION. *Let X be a compact weakly almost complex manifold. Then for every $d \in H^2(X, \mathbf{Z})$, the number $T(X, d)$ is an integer.*

3. Integrality theorems for differentiable manifolds. For the definition of $\hat{A}(X, d)$ and $\hat{A}(X, d, \eta)$ we refer to [1, §§ 25.4, 25.5].

3.1. THEOREM. *Let X be a compact oriented differentiable manifold and d an element of $H^2(X, \mathbf{Z})$ whose restriction* mod 2 *is equal to $w_2(X)$. Then $\hat{A}(X, \frac{1}{2}d)$ is an integer.*

We use the notation of 25.4. Thus E/T is an almost complex manifold, $\pi\colon E/T \to X$ a fibre map, ξ the tangent bundle to X, and $x_1, \cdots, x_q \in H^2(E/T)$ are the roots of the Chern polynomial of a complex structure of $\pi^*(\xi)$. This implies that

$$\pi^*(w_2) = x_1 + \cdots + x_q \mod 2.$$

Furthermore, we have the equality

$$(1) \qquad \hat{A}(X, \tfrac{1}{2}d) = T(E/T, \tfrac{1}{2}(\pi^*(d) - (x_1 + \cdots + x_q))).$$

If now $d \equiv w_2 \bmod 2$, then $\pi^*(d) - (x_1 + \cdots + x_q) \equiv 0 \bmod 2$, therefore the real cohomology class $\frac{1}{2}(\pi^*(d) - (x_1 + \cdots + x_q))$ comes from an integral class under the coefficient homomorphism $\mathbf{Z} \to \mathbf{R}$. Theorem 3.1 follows then from (1) and 2.6.

3.2. COROLLARY. *Let X be a compact, oriented, differentiable manifold with vanishing second Stiefel-Whitney class. Then the genus $\hat{A}(X)$, belonging to the power series $\frac{1}{2}z^{\frac{1}{2}}/\sinh\frac{1}{2}z^{\frac{1}{2}}$, is an integer.*

In this case, we can replace d by 0 in 3.1. Since $\hat{A}(X,0) = \hat{A}(X)$, the corollary follows.

3.3. *Examples.* The polynomial $\hat{A}_1$ is equal to $-p_1/24$. Thus, if $\dim X = 4$,

$$\hat{A}(X, d/2) = ((1 + d/2 + d^2/8)(1 - p_1/24))[X] = (d^2/8 - p_1/24)[X].$$

Since $p_1[X] = 3 \cdot \tau$, where τ is the index of X (see [5, § 0.7; 11, Cor. IV.13]), we get on a 4-dimensional oriented manifold the congruence

$$d^2[X] \equiv \tau \bmod 8 \,(d \in H^2(X, \mathbf{Z}), d \equiv w_2 \bmod 2).$$

This can also be formulated as a statement on the quadratic form of the manifold (F. Hirzebruch-H. Hopf, *Mathematische Annalen*, vol. 136 (1958), pp. 156-172).

Let now X be 6-dimensional. Then

$$\hat{A}(X, d/2) = ((1 + d/2 + d^2/8 + d^3/48)(1 - p_1/24))[X]$$

yields the congruence

$$d^3[X] \equiv (d \cdot p_1)[X] \bmod 48, \quad (d \in H^2(X, \mathbf{Z}), \ d \equiv w_2 \bmod 2).$$

3.4. The coefficient of p_k in $\hat{A}_k$ is $-B_k/((2k!) \cdot 2)$, where B_k is the k-th Bernouilli number [5, p. 13]. This can be easily deduced from [5, § 1]. In fact, by the usual expression of the Pontrjagin classes in terms of Chern classes [5, p. 12], $p_k = (-1)^k 2 \cdot c_{2k}$, modulo decomposable elements; since $A_k = 2^{4k}\hat{A}_k$, formula (12) of [5, p. 15] shows that the coefficient of p_k is $(-1)^k/2$ times the coefficient of c_{2k} in T_{2k}; but this coefficient is also the coefficient of $c_1{}^k$ in T_{2k} [5, Bemerkung 2, p. 15], and it follows readily from the first formula in [5, § 1.7, p. 15] that the latter is equal to $(-1)^{k-1}B_k/(2k)!$, whence our assertion. Together with 3.2, it implies:

3.5. THEOREM. *Let X be a compact oriented differentiable manifold*

of dimension $4k$ whose tangent bundle is trivial when restricted to the complement of some point in X. Then $B_k \cdot p_k[X]/((2k)! \cdot 2)$ is an integer.

This theorem has found an interesting application to the stable homotopy groups of spheres [7]. (See also [6].)

3.6. THEOREM. *Let X be a compact oriented differentiable manifold, η a complex vector bundle over X, and d an element of $H^2(X, \mathbf{Z})$ whose restriction* mod 2 *is equal to $w_2(X)$. Then $\hat{A}(X, \frac{1}{2}d, \eta)$ is an integer.*

We follow the notation of [1, §25.5]. Thus σ is the tangent bundle to X, E/T is the total space of a bundle ξ over X with fibre map π, and $a_j \in H^2(E/T, \mathbf{Z})$ $(1 \leqq j \leqq m)$ are cohomology classes whose sum a is the first Chern class of the complex vector bundle along the fibres. Therefore a, reduced mod 2, is the second Stiefel-Whitney class of the bundle along the fibres $\hat{\xi}$. Since the tangent bundle to E/T is the sum of $\pi^*(\sigma)$ and of $\hat{\xi}$, [1, §7.6], we have

$$\pi^*(d) + a \equiv w_2(E/T) \bmod 2. \tag{2}$$

In [1, §25.5] it is proved that

$$\hat{A}(X, \tfrac{1}{2}d, \eta) = \sum_i \hat{A}(E/T, \tfrac{1}{2}(\pi^*(d) + 2x_i + a), \qquad (x_i \in H^2(X, \mathbf{Z})).$$

Therefore, 3.6 follows from 3.1 and (2).

3.7. *Applications.* Theorem 3.6 gives a positive answer to conjecture (1) in [1, §25.6]. Conjecture (2) ($\hat{A}(X)$ is *even* if $w_2(X) = 0$ and $\dim X \equiv 4 \bmod 8$) and also 3.6 have since been proved by a quite different method [6] which uses Bott's results [3] instead of Milnor's theorem. As a consequence, in 3.5, $B_k p_k[X]$ divided by $(2k)! \cdot 2$ is an even integer for odd k, which yields a slight sharpening of the Kervaire-Milnor theorem [7].

In [1, §26.10] we mentioned the *theorem of Bott* [3] that the Chern class of a complex vector bundle over $\mathbf{S}_{2q}$ is divisible by $(q-1)!$, which had been proved in [1, §25.8] only "exc 2." This and the corresponding divisibility property of Pontrjagin classes follow now from 3.6, in the same way as [1, §§25.8, 25.9] were derived from [1, §25.5].

Let X be a compact almost complex manifold η a complex vector bundle over X. Then $T(X, \eta) = \hat{A}(X, \frac{1}{2}c_1, \eta)$. Therefore by 3.6, $T(X, \eta)$ *is an integer.* This gives an affirmative answer to the first question of Problem 22 in [4]. It follows also that all numbers introduced in connection with the Riemann-Roch theorem, which were proved to be integers "exc 2" in [1, §25.6], are actually integers.

4. Some properties of roots of compact Lie groups. G will always be a compact connected semi-simple Lie group, T a maximal torus. No distinction is made between a point in the universal covering V of T and its image in T (or equivalently, between a point in the Lie algebra $\mathfrak{t}$ of T and its image in T under the exponential map); expressions like positive Weyl chamber, dominant root, simple roots are always understood with respect to some ordering. For the notation see [1, § 2].

4.1. If a, b are roots, the number $q(a,b)=2(a,b)(b,b)^{-1}$ is an integer, and $0 \leqq q(a,b)\cdot q(b,a) \leqq 3$. Consequences: $q(a,b)=0,1,2,3$. If $(aa)<(bb)$ and $(ab)\neq 0$, then $q(ab)=\pm 1$. If $q(a,b)=\pm 1$, then $|q(a,b)| \leqq |q(b,a)|$, hence $(a,a)\leqq(b,b)$; if $(a,b)\neq 0$, and $(a,a)\geqq(b,b)$, then $(a,a)(b,b)^{-1}=1$, 2,3. Let now G be simple. Then $W(G)$ is irreducible. Since the roots of a given length span a subspace invariant under $W(G)$, it follows that if G has a root of a certain length λ, then for any root a of G, there exists a root of length λ not orthogonal to a. Consequently, if the roots are normalized so that the minimal root length is one, then the other possible values are 2, 3. More precisely, it is known that the values of (a,a) are 1, 3 for $G=\boldsymbol{G}_2$, 1, 2 for $G=\boldsymbol{B}_n$, $\boldsymbol{C}_n$, $\boldsymbol{F}_4$ and 1 in the other cases. We recall that if a, b are roots, then so is $c=a-q(a,b)b$, and clearly $(a,a)=(c,c)$.

4.2. *Let G be simple, a_i $(1\leqq i\leqq l)$ be a system of simple roots, and $d=d_1a_1+\cdots+d_la_l$ be the highest root. Then d has maximal length.*

For completeness, we give a proof. There exists a positive root of maximal length $c=c_1a_1+\cdots+c_la_l$ not orthogonal to d (4.1). If $c=d$, we are done, so assume $d\neq c$. Since d is dominant, $c+d$ is not a root, therefore $(c,d)>0$, $q(c,d)=k\geqq 1$, and $c-kd$ is a root. The coefficient of a_i in $c-kd$ must then be smaller in absolute value than d_i, whence $k=1$; but then $(c,c)\leqq(d,d)$ by 4.1 and d has maximal length.

4.3. Let G be simple, a_i $(1\leqq i\leqq l)$ be simple roots of G, $d=d_1a_1+\cdots+d_la_l$ be the highest root, and U be a maximal connected semisimple subgroup containing T. Then there exists an index j such that d_j is prime, U is the centralizer of the point P_j defined by $d_j\cdot a_i(P_j)=\delta_{ij}$ $(1\leqq i\leqq l)$. The simple roots of U with respect to a suitable ordering are the a_i's $(i\neq j)$ and $-d$. The roots of U are exactly the roots of G in which a_j has coefficient 0 or $\pm d_j$. They form a *closed* system (i. e. if a, b are 2 roots of U such that $a+b$ is a root of G, then $a+b$ is a root of U). If the center of G reduces to the identity, then P_j generates a cyclic group of order d_j which is the center

of U. One obtains in this way all maximal connected semi-simple subgroups of maximal rank, up to inner automorphisms. For all this, see [2].

4.4. G being again semi-simple, let a_i $(1 \leqq i \leqq l)$ be its simple roots. Then the equations $a_1 = \cdots = a_l$ define a 1-dimensional subspace contained in the positive Weyl chamber, or also a 1-dimensional torus S in T, to be called the *main diagonal*. It belongs to a three dimensional simple group H, the *principal subgroup* of G in the sense of de Siebenthal, which is defined up to inner automorphisms by those conditions [10, § 13, Th. 2]. H is not contained in a proper subgroup of rank l [10, § 12, Th. 1].

4.5. Let $G = \boldsymbol{SU}(2)$, and Γ_n be the natural representation of degree n of G in the space of homogeneous forms of degree $n-1$ in two variables. As is known, Γ_n is up to equivalence, the only irreducible representation of degree n of G. If n is odd, it is equivalent to a real representation and not faithful. For n even, Γ_n is faithful, equivalent to the complex conjugate representation but not to a real representation. This implies in particular: if Γ is a real representation of G whose restriction to a maximal torus does not contain the trivial representation, then Γ is faithful, breaks up in a sum of real irreducible representations each of which is complex equivalent to a sum $\Gamma_n + \Gamma_n$, n even, hence $\Gamma = \Delta + \Delta$, where Δ is a sum of representations Γ_n, n even.

4.6. THEOREM. *Let U be a proper connected semi-simple subgroup of G of maximal rank. Then the main diagonal of U is singular in G* [10, § 8, Théorème 7].

We may assume that the center of G is reduced to the identity. If G is a direct product $G_1 \times G_2$, then $U = U_1 \times U_2$, where $U_i = U \cap G$ is a subgroup of maximal rank of G_i (see e.g. [2]), and its main diagonal clearly projects onto the main diagonal of U_i $(i = 1, 2)$. Using this and induction, the proof of 4.6 is easily reduced to the case where G is simple, with center reduced to (e), and U is maximal connected. Assuming this from now on, we follow the notation of 4.3 admitting moreover the simple roots to be numbered in such a way that $j = 1$. Let c^* be the point of S defined by $a_i(c^*) = 1$ $(2 \leqq i \leqq l)$ and $d(c^*) = -1$. Then

$$d_1 a_1(c^*) = -1 - d_2 - \cdots - d_l. \tag{1}$$

$a(c^*)$ is integral for all roots a of a system of simple roots of U, hence for all roots of U; therefore, c^* is an element of the center of U.

Assume now that, contrary to our assertion, S is regular. Then $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{u}}S$ does not contain the trivial representation. Let H be the principal subgroup of U containing S. By 4.5, $H = \boldsymbol{SU}(2)$, $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{u}}H$ is faithful, and is a sum of two equivalent representations. From this, and from standard facts about representations of the circle group, it follows that given a complementary root a, there exists a positive complementary root $b \neq \pm a$ such that $b(s) = \pm a(s)$ for all $s \in S$. In particular, taking $a = a_1$, there exists a positive root $b = b_1a_1 + \cdots + b_la_l$ not proportional to a_1, such that

$$b_1a_1(c^*) + b_2 + \cdots + b_l = \pm a_1(c^*) \tag{2}$$

($b_i \geqq 0$, $(i \geqq 1)$, $b_1 \geqq 1$, $(b_2, \cdots, b_l) \neq (0, \cdots, 0)$). Let z be the element $\neq e$ in the center of H. Its connected centralizer U_1 in G is $\neq G$, since G has no center, and contains H, T; the last assertion of 4.4 shows then that U_1 contains U, hence is equal to U, since the latter is maximal connected. Thus, by 4.3, $d_1 = 2$ and $b_1 = 1$. It follows that in the right hand side of (2) we must have the minus sign, and we get

$$-2a_1(c^*) = b_2 + \cdots + b_l, \tag{3}$$

but this, together with $b_i \leqq d_i$, obviously contradicts (1). Therefore S is singular.

4.7. There exists therefore a positive complementary root $b = b_1a_1 + \cdots + b_la_l$ such that $b(s) = 0$ for all $s \in S$. In particular, $b_1a_1(c^*) = -(b_2 + \cdots + b_l)$ is integral; since $0 < b_1 < d_1$ and d_1 is prime, this and (1) show that $a_1(c^*)$ is integral. We have proved:

COROLLARY. *We keep the notation of 4.3 and assume d_j to be prime. Then $1 + d_1 + \cdots + d_l$ is divisible by d_j. If c is the linear form defined by $(a_i, c) = 1$ $(i \neq j)$, $(d, c) = -1$, then (a, c) is integral for all the roots a of G.*

Before stating our next theorem, we discuss some more properties of roots.

4.8. *Let G be simple, a_i $(1 \leqq i \leqq l)$ be the simple roots, and $c = c_1a_1 + \cdots + c_la_l$ be a root of G. Then $c_i(a_i, a_i) \cdot (c, c)^{-1}$ is an integer $(1 \leqq i \leqq l)$.*

It is known that if we perform an inversion with respect to a sphere of radius $2^{\frac{1}{2}}$ in V, then a system of roots is transformed into a system of roots (of a group G' which may or may not be isomorphic to G). Let $e \to \bar{e}$ be this transformation. Then $\bar{e} = 2e \cdot (e, e)^{-1}$, and in particular

$$\bar{c} = 2 \cdot (c, c)^{-1} \sum_i c_i(a_i, a_i) \cdot 2^{-1} \cdot \bar{a}_i,$$

$$\bar{c} = \sum_i c_i((a_i, a_i) \cdot (c, c)^{-1}) \bar{a}_i.$$

This shows first that all roots in the new system are linear combinations with coefficients of the same sign of $\bar{a}_1, \cdots, \bar{a}_l$; hence $\bar{a}_i > 0$ defines a Weyl chamber for the new system, and the $\bar{a}_i$ are a simple system of roots. Therefore the coefficients of $\bar{c}$ are integers.

4.9. *Let G be simple, U be a maximal connected subgroup of maximal rank, and b be the sum of the positive roots of U. Then (b, a) is integral for all roots a of G, the minimal root length being assumed to be* 1.

Proof. Since $W(U) \subset W(G)$, it is enough to prove this for one particular ordering. Let us consider one, say $\mathcal{S}$, with respect to which the simple roots of U are, in the notation of 4.3, $-d$ and the a_i's $(i \neq j)$. We have then [1, § 3.1]

$$(4) \qquad (b, a_i) = (a_i, a_i) \quad (i \neq j), \quad (d, b) = -(d, d).$$

The minimal root length being assumed to be 1, these are all integers (4.1) and it is therefore sufficient to show that (b, a_j) is an integer. (4) yields

$$(5) \qquad d_j(b, a_j) = -(d, d) - \sum_{i \neq j} d_i(a_i, a_i),$$

hence $d_j(b, a_j)$ is an integer. By 4.1, (b, a_j) is at any rate a half integer, so that we are done if d_j is odd. If all scalar products (a, a) are equal to 1, our assertion follows from (4) and 4.7; there remains therefore the case where $d_j = 2$ and (4.1) there are two root lengths. By 4.8, $d_i(a_i, a_i)(d, d)^{-1}$ is integral and by 4.2, d has maximal length; thus if $(d, d) = 2$, each term on the right hand side of (5) is even, and (b, a_j) is integral. If now $(d, d) = 3$, then $G = \boldsymbol{G}_2$, $d = 3a_1 + 2a_2$, which implies $j = 2$, $(a_1, a_1) = 1$ and $d_2(b, a_2) = -6$.

4.10. *Let G be simple, and assume that there are two root lengths $s < t$. Then any root of length t is the sum of two roots of length s.*

Let a be a root of length t. Since $W(G)$ is irreducible, there exists at least one root b of length s, not orthogonal to a; then $c = b - q(b, a)a$ is a root of length s (4.1). Since $q(b, a) = \pm 1$ by 4.1, our assertion is proved.

4.11. THEOREM. *Let U be a proper connected semi-simple subgroup of maximal rank of G and let b be the sum of the positive roots of U. Then b is singular in G.*

As in 4.6, it is first seen that it suffices to prove our assertion for some ordering, when G is simple and U is maximal connected.

Proof will be by contradiction. Assume that b is regular. Let then $\mathcal{S}$ be the ordering of the roots of G defined by $a > 0$ if and only if $(b, a) > 0$ [1, § 2.8]. On the roots of U, it coincides with the original ordering, with respect to which b had been defined, as follows from [1, § 3.1], hence b is also the sum of the roots of U which are positive for $\mathcal{S}$. Let us number the simple roots a_i $(1 \leqq i \leqq l)$ for $\mathcal{S}$ so that a_i is a root of U if and only if $i \leqq j$. (A priori, it is conceivable that no a_i belongs to U, in which case we set $j = 0$.) The $l - j$ other simple roots of U will then be denoted by a_i' $(j+1 \leqq i \leqq l)$. Of course $j \neq l$ since $U \neq G$. By [1, § 3.1]

$$(6) \qquad (b, a_i) = (a_i, a_i) \quad (i \leqq j), \qquad (b, a_i') = (a_i', a_i') \quad (i \geqq j+1).$$

For a given a_i', there exist non negative integral c_j's such that $a_i' = c_1 a_1 + \cdots + c_l a_l$, therefore

$$(7) \qquad \begin{aligned} (a_i', a_i') = (b, a_i') = c_1(a_1, a_1) + \cdots + c_j(a_j, a_j) \\ + c_{j+1}(b, a_{j+1}) + \cdots + c_l(b, a_l). \end{aligned}$$

At least two c_i's are not zero; by the definition of $\mathcal{S}$ and 4.9 we have $(a_i', a_i') \geqq 2$, hence a_i' has maximal length.

We want to prove now that $G \neq \boldsymbol{G}_2$. If G were equal to $\boldsymbol{G}_2$, then $a_i' = c_1 a_1 + c_2 a_2$ with $c_1 \cdot c_2 \neq 0$, $(a_i', a_i') = 3$, hence by 4.8 the coefficient of the root of length one would have to be a multiple of 3, but this contradicts (7) and the fact that all scalar products are integers $\geqq 1$.

Thus, $G \neq \boldsymbol{G}_2$, there are two root lengths, and $(a_i', a_i') = 2$. It also follows that two of the c_i's, say $c_{s(i)}, c_{t(i)}$ $(s(i) < t(i))$ are equal to 1, and the others to zero. Since a_i' is simple as a root of U, we must have $t(i) \geqq j+1$, and then also $s(i) \geqq j+1$ since the root system of U is closed (4.3); (4.8) implies then that $(a_k, a_k) = 2$ for $k = s(i), t(i)$. In particular, we see that all simple roots of G of length one belong to U.

Let now c be the first (with respect to $\mathcal{S}$) positive complementary root of length one. This exists in view of 4.10. In order to have a contradiction, it is enough to prove that $(b, c) \leqq 0$ and this will follow if we show that

$$(8) \qquad (c, a_i) \leqq 0 \quad (i \leqq j), \qquad (c, a_i') \leqq 0 \quad (i \geqq j+1).$$

By the above, c is not a simple root, therefore $c - q(c, a_i) a_i$, expressed as linear combination of the a_i $(1 \leqq i \leqq l)$, has some positive coefficient. If $(c, a_i') \neq 0$, then $q(c, a_i') = \pm 1$ by 4.1 and because of $(c, c) = 1$, $(a_i', a_i') = 2$; since

a_i' is the sum of two simple roots, it follows again that $c - q(c, a_i')a_i'$ has some positive coefficient. Therefore, the roots $c - q(c, a_i)a_i$ $(i \leqq j)$, and $c - q(c, a_i')a_i'$ $(i \geqq j+1)$ are positive, and moreover complementary of length one since c is. By the choice of c, they must then be greater than c, in the sense of $\mathcal{S}$, and this implies (8).

5. The Stiefel-Whitney class of G/T.

5.1. Let ξ be a differentiable bundle with connected fibres. Let $b \in B_\xi$ and $F = \pi_\xi^{-1}(b)$. The normal bundle to F in E_ξ is of course trivial, since it is induced by π_ξ from the tangent space of B_ξ at b. Therefore F is orientable if E_ξ is. Furthermore, the multiplication theorem [1, § 9.7] shows that $w(F)$ (resp. $\tilde{p}(F)$) is the restriction to F of $w(E_\xi)$, (resp. $\tilde{p}(E_\xi)$). In particular, it reduces to 1 if E_ξ is parallelizable. More precisely, if the S-class (§ 1) of the tangent bundle to E_ξ is zero, then the S-class of the tangent bundle to F is also zero. A similar observation is valid for Chern classes and S-equivalence class in a complex analytic (or almost complex) bundle.

5.2. Assume now that ξ is a principal differentiable bundle. The bundle along the fibres $\hat{\xi}$ [1, § 7.4] is then parallelizable. Since the tangent bundle to E_ξ is the sum of $\hat{\xi}$ and of the bundle induced by π_ξ from the tangent bundle to B_ξ [1, § 7.6], its S-class will be zero if the S-class of the tangent bundle to B_ξ is zero. Furthermore, the multiplication theorem gives

$$\pi_\xi^*(w(B_\xi)) = w(E_\xi), \qquad \pi_\xi^*(\tilde{p}(B_\xi)) = \tilde{p}(E_\xi).$$

Hence if $w(B_\xi) = 1$ (resp. $\tilde{p}(B_\xi) = 1$), then $w(E_\xi) = 1$ (resp. $\tilde{p}(E_\xi) = 1$). If $w(E_\xi) = 1$ (resp. $\tilde{p}(E_\xi) = 1$), and π_ξ^* is injective, then $w(B_\xi) = 1$ (resp. $\tilde{p}(B_\xi) = 1$). (Coefficients in a field of characteristic two for the Stiefel-Whitney classes, arbitrary coefficients for the Pontrjagin classes.) Again a similar assertion is valid for Chern classes in the almost complex case.

5.3. PROPOSITION. *Let G be a compact connected Lie group, S a toral subgroup. Then $w(G/S) = 1$ and $\tilde{p}(G/S) = 1$.*

Let T be a maximal torus containing S. Then we have the principal fibering $(G/S, G/T, T/S)$, therefore (5.2) it is enough to prove 5.3 for $S = T$. For the Pontrjagin class, see [1, § 10.9]. There remains to prove that $w(G/T) = 1$. Without loss of generality it may be assumed that G is semi-simple and simply connected. Let Q be the subgroup of elements of order two in T. Then $(G/Q, G/T, T/Q, \pi)$ is a principal fibering. The total space, being the quotient of a group by a finite subgroup, is parallelizable, therefore,

(5.2) it will be enough to show that π^* is injective in cohomology mod 2. Since G and G/T are simply connected, $\pi_1(G/Q)=Q$, and the map $\pi_1(T/Q)\to\pi_1(G/Q)$ defined by the inclusion i is surjective. It follows easily that $i^*: H^*(G/Q,\mathbf{Z}_2)\to H^*(T/Q,\mathbf{Z}_2)$ is an isomorphism in dimension 1. But T/Q is a torus, hence $H^*(T/Q,\mathbf{Z}_2)$ is generated by its element of degree $\leqq 1$; therefore i^* is surjective, the fibre is totally non homologous to zero in cohomology mod 2; as is well known, this implies that π^* is injective.

5.4. It can also be derived directly from 5.1, 5.2 that *the S-class* (§ 1) *of the tangent bundle to* G/S (S *toral subgroup of* G) *is zero.*

In view of 5.2 and of the existence of the principal fibering (G/S, G/T, T/S) it is enough to prove 5.4 when $S=T$ is a maximal torus. Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{R}$ the set of regular elements, and let G operate on $\mathfrak{g}$ by the adjoint representation. Since the centralizer of a regular element $x\in\mathfrak{g}$ is the maximal torus containing the one-parameter subgroup spanned by x, the orbits of G in $\mathfrak{R}$ are homeomorphic to G/T. Moreover, it is classical, and easily checked, that these orbits are the fibres in a differentiable fibering of $\mathfrak{R}$. Since $\mathfrak{R}$ is parallelizable, as an open subset of $\mathfrak{g}$, our assertion follows from 5.1.

The nullity of $w_2(G/T)$ was noticed in [1, § 22.3] and, as remarked above, $\tilde{p}(G/T)=1$ was also proved in [1, § 10.9].

5.5. Without entering into details, let us mention a case containing the preceding one, in which 5.1 applies. Let G operate differentiably on a connected manifold M. Among the different stability groups $G_x=\{g\in G, g\cdot x=x\}$, let H be one of smallest dimension, which has the minimal number of connected components among stability groups of that dimension. Then the set of points whose stability group is conjugate to H is an open set in M, which is differentiably fibered by the orbits [9, pp. 221-222]. Those are homeomorphic to G/H, and are called the *main orbits.* 5.1 yields then the

PROPOSITION. *Let G be a compact Lie group acting on a connected manifold M, and let F be a main orbit. Then F is orientable if M is. $w(F)$ and $\tilde{p}(F)$ are the restrictions to F of $w(M)$ and $\tilde{p}(M)$. If the S-class of the tangent bundle to M is zero, then the S-class of the tangent bundle to F is zero. In particular, if G/H is homeomorphic to the main orbit of a linear representation then it is orientable, the S-class of its tangent bundle is zero, and $w(G/H)=1$, $\tilde{p}(G/H)=1$.*

The proof given in 5.4 is the particular case of 5.5 corresponding to the adjoint representation, where the main orbits are homeomorphic to G/T.

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ON THE COBORDISM RING Ω^* AND A COMPLEX ANALOGUE, PART I.*

By J. Milnor.

This paper will prove that the cobordism groups Ω^i, defined by Thom [15], have no odd torsion.[1] Furthermore, it is shown that certain related groups $\pi_{i+2n}M(U_n)$ have no torsion at all; providing that n is large. The proofs are based on a spectral sequence due to J. F. Adams [1, 2].

The following is a brief summary of Thom's constructions. Let G be a subgroup of the orthogonal group O_n. (More generally one could start with any Lie group G, together with a specified representation into O_n.) Beginning with a universal bundle for G we can form:

1) The weakly associated bundle having the disk D^n as fibre. Let $\pi\colon E \to B(G)$ denote the projection map of this bundle.

2) The weakly associated bundle having the sphere S^{n-1} as fibre. Let $\partial E \subset E$ denote the total space.

The *Thom space* $M(G)$ is now defined as the identification space obtained from E by collapsing ∂E to a point.

Taking G to be the rotation group $SO_n \subset O_n$, Thom showed that the homotopy group $\pi_{i+n}M(SO_n)$ is independent of n, providing that n is large. He showed that this group is isomorphic to the "cobordism group" Ω^i; and determined its structure up to torsion. The 2-torsion subgroup of Ω^i has recently been determined by C. T. C. Wall. Hence the assertion that Ω^i has no odd torsion completes the description of this group.

Let $M(U_n)$ denote the Thom space for the unitary group $U_n \subset O_{2n}$. In Part II of this paper it will be shown that the stable homotopy group $\pi_{i+2n}M(U_n)$ can be interpreted as a "complex cobordism group." Part I will determine the structure of this group without attempting to interpret it.

* Received July 27, 1959.

[1] Added in proof. This result has been obtained independently by B. G. Averbuch, *Doklady Akademii Nauk SSSR*, vol. 125 (1959), pp. 11-14. The results on complex cobordism have been obtained independently by Novikov.

The first section proves several lemmas concerning the Steenrod algebra, which are needed later. The second section describes the Adams spectral sequence, which relates the cohomology module of any space to the stable homotopy groups of the space. Sections 3 and 4 complete the argument by computing the cohomology modules of $M(U_n)$ and $M(SO_n)$ respectively.

1. Lemmas concerning the Steenrod algebra. Let A denote the Steenrod algebra corresponding to a fixed prime p. (See Cartan [6], Adem [3].) The Bochstein coboundary operation will be denoted by $Q_0 \in A^1$. The two-sided ideal generated by Q_0 in A will be denoted by (Q_0).

LEMMA 1. *The Steenrod algebra contains a subalgebra A_0 with the following properties.*

(i) *A_0 is a Grassmann algebra over Z_p with generators $Q_0, Q_1, \cdots$ of odd dimension.*

(ii) *A is free as a right A_0-module.*

(iii) *The identity map of A induces an isomorphism between the left A-modules $A \otimes_{A_0} Z_p$ and $A/(Q_0)$.*

[*Explanation of* (iii). The field Z_p is considered as a left A_0-module with $Q_i Z_p = 0$. Hence $A \otimes_{A_0} Z_p$ is the quotient of A by the left ideal $AQ_0 + AQ_1 + AQ_2 + \cdots$.]

Proof for the case p odd. We will first prove the corresponding statements with left and right interchanged. According to Milnor [10, Theorem 4a]:

(1) There is a basis for A over Z_p consisting of elements $Q_0^{e_0} Q_1^{e_1} \cdots \mathfrak{P}^R$. Here the integers $e_0, e_1, \cdots$ should be 0 or 1, and almost all zero. The letter R stands for a sequence $(r_1, r_2, \cdots)$ of non-negative integers, almost all zero.

[*Explanation.* The element $\mathfrak{P}^R$ is a complicated polynomial in the Steenrod operations, with dimension $\sum r_j(2p^j - 2)$. For the special case $R = (r, 0, 0, 0, \cdots)$ the element $\mathfrak{P}^R$ is equal to the Steenrod operation $\mathfrak{P}^r$. The element Q_i of dimension $2p^i - 1$ can be defined inductively by the rule $Q_{i+1} = \mathfrak{P}^{p^i} Q_i - Q_i \mathfrak{P}^{p^i}$.]

Furthermore:

(2) The elements Q_i are odd dimensional, and satisfy $Q_i Q_j + Q_j Q_i = 0$, $Q_i Q_i = 0$.

Thus the Q_i generate a Grassmann algebra which may be denoted by $A_0 \subset A$. Clearly A is free as a left A_0-module, with basis $\{\mathfrak{P}^R\}$.

Consider the right ideal $Q_0A + Q_1A + Q_2A + \cdots$. The following identity (see [10, Theorem 4a]) proves that this is also a left ideal. Define $p^i\Delta_j$ as the sequence $(0, \cdots, 0, p^i, 0, \cdots)$ with p^i in the j-th place.

(3) $\mathfrak{P}^R Q_i$ is equal to $Q_i \mathfrak{P}^R + \sum Q_{i+j} \mathfrak{P}^{R-p^i\Delta_j}$, to be summed over all $j > 0$ for which $R - p^i\Delta_j$ is a sequence of non-negative integers. (That is, all j for which $r_j \geqq p^i$.)

Thus $Q_0A + Q_1A + \cdots$ is a two-sided ideal which contains Q_0, and therefore contains (Q_0).

As a special case of (3), the identity $\mathfrak{P}^{\Delta_j} Q_0 = Q_0 \mathfrak{P}^{\Delta_j} + Q_j$ is valid. Thus the elements Q_j belong to the ideal (Q_0). This proves that the ideal $Q_0A + Q_1A + \cdots$ is equal to (Q_0). Dividing A by these ideals, it follows that $Z_p \otimes_{A_0} A$ is isomorphic to $A/(Q_0)$.

This proves Lemma 1 for p odd, except that right and left have been interchanged. To complete the proof it is only necessary to recall:

(4) There exists an anti-automorphism of A; that is, a Z_p-isomorphism $c\colon A \to A$ satisfying

$$c(xy) = (-1)^{\dim x \dim y} c(y) c(x).$$

Furthermore, c carries Q_i into $-Q_i$.

This is proved in [10, § 7]. Clearly Lemma 1 follows (for p odd).

LEMMA 2. *The elements $\mathfrak{P}^R \in A$ yield a basis over Z_p for the quotient algebra $A/(Q_0)$.*

Proof for p odd. Recall that $\{\mathfrak{P}^R\}$ forms a basis for A, considered as a left A_0-module. Hence it forms a basis for $Z_p \otimes_{A_0} A = A/(Q_0)$ over Z_p, which completes the proof.

Conventions. The sum $R + R'$ of two sequences is defined as the term by term sum, and nR denotes the sequence $(nr_1, nr_2, \cdots)$. The binomial coefficient (R, R') is defined as the product over i of $(r_i + r'_i)!/r_i!r'_i!$. The symbol Δ_j stands for a sequence with 1 in the j-th place and zero elsewhere.

Proof of Lemmas 1 and 2 for the case $p = 2$. The Steenrod algebra over Z_2 has a basis consisting of elements Sq^R of dimension $r_1 + 3r_2 + 7r_3 + \cdots$. (See [10, Appendix 1].) Define $\mathfrak{P}^R$ to be Sq^{2R} and define Q_{i-1} to be Sq^{Δ_i}. (For example $Q_0 = \mathrm{Sq}^{\Delta_1} = \mathrm{Sq}^1$ which checks with the definition of Q_0 as the

Bochstein coboundary operator.) Then we will prove Assertions (1), (2), (3) and (4) above. Using these, the proof of Lemmas 1 and 2 can be carried out just as for p odd.

The formula for products $\mathrm{Sq}^{R'}\mathrm{Sq}^{R}$ is rather complicated; however the following special case will suffice.

(5) If E is a sequence satisfying $e_i \leqq 1$, then $\mathrm{Sq}^E\mathrm{Sq}^R$ is equal to $(E,R)\,\mathrm{Sq}^{E+R}$.

For a proof see [10, Corollary 4 and Appendix 1]. As examples, taking $E=\Delta_{i+1}$, $R=\Delta_{j+1}$ we find that $Q_iQ_j=Q_jQ_i$, and that $Q_iQ_i=0$. This proves Assertion (2) for the case $p=2$.

By induction the product $Q_0^{e_1}Q_1^{e_2}\cdots$ is equal to Sq^E. Furthermore, a binomial coefficient of the form $(E,2R)$ is always odd, hence $\mathrm{Sq}^E\mathcal{P}^R=\mathrm{Sq}^E\mathrm{Sq}^{2R}$ is equal to Sq^{E+2R}. Since every sequence can be written uniquely in the form $E+2R$, it follows that these elements form a basis for A over Z_2. This proves Assertion (1).

Proof of Assertion (3) *for* $p=2$. Direct application of the general product rule [10, Theorem 4b] shows that

$$\mathrm{Sq}^{2R}\mathrm{Sq}^{\Delta_{i+1}}=\mathrm{Sq}^{\Delta_{i+1}}\mathrm{Sq}^{2R}+\sum \mathrm{Sq}^{2R-2^{i+1}\Delta_j+\Delta_{i+1+j}},$$

to be summed over all $j\geqq 1$ for which $r_j\geqq 2^i$. On the other hand, using Assertion (5), the j-th term on the right can be written as

$$\mathrm{Sq}^{\Delta_{i+1+j}}\mathrm{Sq}^{2R-2^{i+1}\Delta_j}=Q_{i+j}\mathcal{P}^{R-2^i\Delta_j}.$$

Thus $\mathcal{P}^RQ_i=Q_i\mathcal{P}^R+\sum Q_{i+j}\mathcal{P}^{R-2^i\Delta_j}$, as required.

Since Assertion (4) is also true for $p=2$, this completes the proof of Lemmas 1 and 2.

[*Remark.* There is one essential difference between the case p odd and the case $p=2$. For p odd the elements $\mathcal{P}^R$ span a subalgebra of A isomorphic to $A/(Q_0)$; but for $p=2$ there is no such subalgebra. This can be seen using the identity $\mathrm{Sq}^2\mathrm{Sq}^2=\mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1\neq 0$.]

The symbol Δ_0 will denote the sequence $(0,0,\cdots)$.

LEMMA 3. *If p is odd, then the cohomology operations $\mathcal{P}^R$ have the following properties.*

(1) *For $x,y\in H^*(X;Z_p)$ the element $\mathcal{P}^R(xy)$ is equal to*

$$\sum_{R_1+R_2=R}(\mathcal{P}^{R_1}x)(\mathcal{P}^{R_2}y).$$

(2) *For a 2-dimensional cohomology class $t \in H^2(X; Z_p)$, the element $\mathcal{P}^R t$ is equal to t^{p^i} if $R = \Delta_i$; and is zero if R is not equal to one of the sequences $\Delta_0, \Delta_1, \Delta_2, \cdots$.*

Proof. The first assertion follows from [10, Lemma 9]. For the special case $R = r\Delta_1$, the second assertion is well known. That is:

$$\mathcal{P}^0 t = t, \quad \mathcal{P}^1 t = t^p, \quad \mathcal{P}^r t = 0 \text{ for } r > 1.$$

But every $\mathcal{P}^R$ is a "polynomial" in the Steenrod operations $\mathcal{P}^r$. Proceeding by induction on the complexity of this polynomial, we see that $\mathcal{P}^R t$ must have the form kt^i, where $k \in Z_p$ is some constant, and $2i$ is the dimension.

To evaluate k it is sufficient to consider one example. As example, let X be the $2i$-skeleton of the Eilenberg-MacLane complex $K(Z_p, 1)$. According to [10, Lemmas 4, 6] we have:

$$\lambda(t) = t \otimes \xi_0 + t^p \otimes \xi_1 + \cdots;$$

hence

$$\mathcal{P}^R t = \sum_i \langle \mathcal{P}^R, \xi_i \rangle t^{p^i}.$$

Using the definition of $\mathcal{P}^R$, this is equal to t^{p^i} if $R = \Delta_i$ and is zero otherwise. This completes the proof.

For the prime $p = 2$, both assertions of Lemma 3 would be false. However the following modified assertions are proved by the same method:

(1′) $\quad \mathrm{Sq}^R(xy) = \sum_{R_1 + R_2 = R} (\mathrm{Sq}^{R_1} x)(\mathrm{Sq}^{R_2} y).$

(2′) If $a \in H^1(X; Z_2)$, then $\mathrm{Sq}^{\Delta_i} a = a^{2^i}$; and $\mathrm{Sq}^R a = 0$ for R not of the form Δ_i.

Using these statement the following result will be proved.

LEMMA 3′. *Let $p = 2$ and let $H^*(X; Z_2)$ be a cohomology ring which is annihilated by the operation $Q_0 = \mathrm{Sq}^1$. The assertions* (1) *and* (2) *of Lemma 3 are valid as originally stated.*

Proof of (1). If R_1 is a sequence containing some odd integer, then Sq^{R_1} belongs to the ideal (Q_0) (compare the proof of Lemma 1), and therefore annihilates the cohomology of X. Thus in formula (1′) above, it is sufficient to consider sequences R_1 and R_2 which are "even." This proves assertion (1).

Proof of (2). It will be convenient to weaken the hypothesis on X, and assume only that $\mathrm{Sq}^1 t = 0$. Then just as in the proof of Lemma 3, it follows

that $\mathcal{P}^R t$ has the form kt^i. In order to determine the constant $k \in Z_2$, it is sufficient to consider the example of a real projective space X, with $t = a^2$, Using (1′) and (2′) it is seen that $\mathcal{P}^R t$ equals t^{2^i} for $R = \Delta_i$ and equals zero otherwise. This completes the proof of Lemma 3′.

2. The spectral sequence of Adams. Let X, Y be finite CW-complexes with base point denoted by o; and let A be the Steenrod algebra for some fixed prime p. Thus the cohomology group $H^*(X \bmod o; Z_p)$ is a graded left A-module.

The m-fold *suspension* $S^m X$ is obtained from the product $X \times I^m$ by collapsing $(X \times \partial I^m) \cup (o \times I^m)$ to a point. Here I^m denotes the unit m-cube. The *stable track group* $\{X, Y\}_n$ is the direct limit under suspension of the group of homotopy classes of maps $S^{m+n} X \to S^m Y$. (The integer n may be positive or negative.)

THEOREM OF ADAMS. *There exists a spectral sequence $\{E_r^{st}, d_r\}$ determined by X, Y and p such that*

$$E_2^{st} = \operatorname{Ext}_A^{st}(H^*(Y \bmod o; Z_p), H^*(X \bmod o; Z_p))$$

and such that

$$E_\infty^{st} = B^{st}/B^{s+1\, t+1},$$

where $\{X, Y\}_n = B^{0\,n} \supset B^{1\, n+1} \supset B^{2\, n+2} \supset \cdots$ is a certain filtration. The intersection $\bigcap_s B^{s\, n+s}$ of these groups is equal to the subgroup of $\{X, Y\}_n$ consisting of elements whose order is finite and prime to p. Each succeeding term E_{r+1} of the spectral sequence is equal to the homology of E_r with respect to the differential operator

$$d_r \colon E_r^{st} \to E_r^{s+r\, t+r-1};$$

and E_∞ is the limit as $r \to \infty$ of E_r.

The functor $\operatorname{Ext}_A^{st}$ is defined as follows. If M and N are graded left A-modules let $\operatorname{Hom}_A{}^t(M, N) = \operatorname{Ext}_A^{0t}(M, N)$ denote the group of A-homomorphisms $M \to N$ of degree $-t$. Choose a projective resolution

$$\cdots \to P_2 \xrightarrow{d} P_1 \xrightarrow{d} P_0 \to M \to 0,$$

where the A-homomorphisms d have degree zero. Then $\operatorname{Ext}_A^{st}(M, N)$ is defined as the homology group (kernel modulo image) of the sequence

$$\operatorname{Hom}_A{}^t(P_{s-1}, N) \xrightarrow{d^*} \operatorname{Hom}_A{}^t(P_s, N) \xrightarrow{d^*} \operatorname{Hom}_A{}^t(P_{s+1}, N).$$

It will be convenient to add an E_1 term to the spectral sequence by defining $E_1^{st} = \mathrm{Hom}_A{}^t(P_s, N)$, $d_1 = d^*$.

For the special case $X = S^0$ this theorem is proved in Adams [1]. The more general case is proved by the same argument. It is only necessary to replace the homotopy group $\{S^0, \ \}_n$ by the track group $\{X, \ \}_n$ throughout. See Adams [2].

More generally the finite complex Y may be replaced by a "spectrum" in the sense of Lima [9] and Spanier [13]; or by an "object in the stable category" in the sense of Adams [2]. For our purpose the following definition will be convenient. A *stable object* $\mathbf{Y}$ is a sequence of CW-complexes $(Y_0, Y_1, \cdots)$ such that each suspension SY_i is a subcomplex of Y_{i+1}. The imbedding $SY_i \subset Y_{i+1}$ must be explicitly given.

Given such an object, define the chain group $C_n(\mathbf{Y})$ as the direct limit under suspension of the chain groups $C_{n+i}(Y_i \operatorname{mod} o)$. Homology and cohomology groups are then defined as usual. Similarly, for any finite complex X define $\{X, \mathbf{Y}\}_n = \text{dir.lim.}\{S^iX, Y_i\}_n$. The abbreviation $\pi_n\mathbf{Y}$ will sometimes be used for $\{S^0, Y\}_n$.

Remark. The suspension homomorphism of chain groups should be defined by the correspondence

$$\alpha \to \alpha \times \iota, \text{ for } \alpha \in C_*(Y_i \operatorname{mod} o), \ \iota \in C_1(I \operatorname{mod} \partial I),$$

so as to commute with boundary homomorphisms.

Examples. Any finite complex Y may be defined with the stable object

$$\mathbf{Y} = (Y, SY, S^2Y, \cdots).$$

We will see later that the suspension of the Thom space $M(SO_n)$ is imbedded naturally as a subcomplex of $M(SO_{n+1})$. Hence the *stable Thom object*

$$\mathbf{M}(SO) = (o, M(SO_1), M(SO_2), \cdots)$$

is defined. Note that the track group

$$\{S^0, \mathbf{M}(SO)\}_n = \text{dir. lim.}\, \pi_{n+i}(M(SO_i))$$

is isomorphic to the cobordism group Ω^n.

Assertion. The theorem of Adams remains valid if the finite complex Y is replaced by any stable object $\mathbf{Y}$; providing that the following *finiteness condition* is satisfied. The groups $C_n(\mathbf{Y}; Z)$ should be finitely generated, and should vanish for n less than some constant.

This can be proved in two ways. One can simply take the direct limit of the spectral sequences for the "finite sub-objects" of $\mathbf{Y}$; or the theorem can be proved from the beginning in the stable category. See Adams [2]. The second approach is preferable, since the proof is much easier in the stable category. Details will not be given.

Using the Adams spectral sequence we will prove the following key result. Let $\mathbf{Y}$ be an object such that $H^n(\mathbf{Y};Z_p)$ is zero for n odd. Then $H^*(\mathbf{Y};Z_p)$ is annihilated by the element Q_0, and hence can be considered as a graded module over the quotient algebra $A/(Q_0)$.

THEOREM 1. *If $H^*(\mathbf{Y};Z_p)$ is a free $A/(Q_0)$-module with even dimensional generators, and if $C_*(\mathbf{Y};Z)$ satisfies the finiteness condition, then the stable homotopy group $\{S^0,\mathbf{Y}\}_n$ contains no p-torsion.*

The idea of the proof is to compute the spectral sequence for the track group $\{X,\mathbf{Y}\}_n$, where X is a "co-Moore space" having cohomology groups $H^i(X \bmod o;Z)$ equal to Z_p for $i=k$ and equal to zero for $i\neq k$.

The following universal coefficient theorem has been proved by Peterson [11]. *There exists an exact sequence*

$$0\to\{S^k,\mathbf{Y}\}_n\otimes Z_p\to\{X,\mathbf{Y}\}_n\to \operatorname{Tor}(\{S^k,\mathbf{Y}\}_{n-1},Z_p)\to 0.$$

An immediate consequence is the following.

LEMMA 4. *If $\{S^0,\mathbf{Y}\}_n$ contains p-torsion, then $\{X,\mathbf{Y}\}_m$ must be non-trivial for two consecutive values of m.*

On the other hand, assuming that $H^*(\mathbf{Y};Z_p)$ is a free $A/(Q_0)$-module on even dimensional generators, we will see that $\{X,\mathbf{Y}\}_m$ is zero for m odd. This will prove Theorem 1.

Construction of an A-free resolution for $H^*(\mathbf{Y};Z_p)$.

First consider the Grassmann algebra A_0 and the A_0-module Z_p. According to Cartan's theory of constructions, to each Grassmann algebra A_0 there corresponds a twisted polynomial algebra P and a differential operator d on $A_0\otimes P$ so that this tensor product becomes acyclic. If A_0 has generators $Q_0, Q_1,\cdots$, then P has a basis over Z_p consisting of elements $b(r_0,r_1,\cdots)$ of dimension $\sum r_i(\dim Q_i+1)$. The integers $r_0,r_1,\cdots$ should be non-negative and almost all zero. The differential operator d is defined as follows. (In order to make the signs come out correctly, we let d act on the right.) For any $a\in A_0$:

$$a\otimes b(r_0,r_1,\cdots)d=\sum aQ_i\otimes b(r_0,\cdots,r_i-1,r_{i+1},\cdots),$$

summed over all i for which $r_i>0$.

Proof that $A_0\otimes P$ is acyclic. For a Grassmann algebra on one generator, see Cartan [5, p. 704, I]. But a Grassmann algebra with finitely many generators in each dimension can be considered as a tensor product of Grassmann algebras with one generator. Hence the conclusion follows by applying the Künneth theorem.

This conclusion can be formulated as follows. Let F_s be the free A_0-module generated by those symbols $b(r_0,r_1,\cdots)$ for which $r_0+r_1+\cdots=s$. Then $A_0\otimes P$ can be considered as the direct sum $F_0+F_1+\cdots$. The augmentation $\epsilon\colon F_0\to Z_p$ is the A_0-homomorphism defined by $b(0,0,\cdots)\to 1$. It follows that *the sequence*

$$\cdots\xrightarrow{d}F_1\xrightarrow{d}F_0\xrightarrow{\epsilon}Z_p\to 0$$

is an A_0-free resolution of Z_p.

Now apply the functor $A\otimes_{A_0}$ to this exact sequence. Since A is free as a right A_0-module, we obtain an exact sequence

$$\cdots\to A\otimes_{A_0}F_1\to A\otimes_{A_0}F_0\to A\otimes_{A_0}Z_p\to 0$$

of left A-modules. Furthermore, each $A\otimes_{A_0}F_s$ is a free A-module. Thus we have constructed an A-free resolution of $A\otimes_{A_0}Z_p$.

According to Lemma 1, the A-module $A/(Q_0)$ is isomorphic to $A\otimes_{A_0}Z_p$. Hence in order to form an A-free resolution of any $A/(Q_0)$-free module, it is sufficient to take the direct sum of a number of copies of the above resolution. This proves the following.

LEMMA 5. *Let $H^*(\mathbf{Y};Z_p)$ be a free module over $A/(Q_0)$ with basis $\{y_\alpha\}$. Then there exists an A-free resolution*

$$\cdots\to F_1'\to F_0'\to H^*(\mathbf{Y};Z_p)\to 0,$$

where each F_s' has a basis consisting of elements $b_\alpha(r_0,r_1,\cdots)$, with $r_0+r_1+\cdots=s$. The dimension of such a basis element is equal to $\dim y_\alpha+\sum 2r_i(p^i-1)+s$.

[*Explanation.* The integer s has been added to the dimension of $b_\alpha(r_0,r_1,\cdots)$ so that the homomorphisms $d'\colon F_s'\to F_{s-1}'$ will have degree zero.]

Now consider the complex X consisting of a circle with a 2-cell attached by a map of degree p. Let

$$x \in H^1(X \bmod o; Z_p), \qquad Q_0 x \in H^2(X \bmod o; Z_p)$$

be generators. Then the term

$$E_1^{st} = \mathrm{Hom}_A{}^t(F_s', H^*(X \bmod o; Z_p))$$

of the spectral sequence for $\{X, \boldsymbol{Y}\}$ has a basis consisting of the following elements.

(1) For each $b_\alpha(r_0, r_1, \cdots)$ of dimension $t+1$, the homomorphism $h_\alpha(r_0, r_1, \cdots)$ which carries this basis element into x and carries the other basis elements into zero.

(2) For each $b_\alpha(r_0, r_1, \cdots)$ of dimension $t+2$, the homomorphism $h_\alpha'(r_0, r_1, \cdots)$ which carries this basis element into $Q_0 x$ and carries the other basis elements into zero.

The boundary operator $d_1\colon E_1^{st} \to E_1^{s+1\,t}$ is given by

$$d_1 h_\alpha(r_0, r_1, \cdots) = h_\alpha'(r_0 + 1, r_1, \cdots),$$

and

$$d_1 h_\alpha'(r_0, r_1, \cdots) = 0.$$

Thus E_2^{st} has as basis the set of elements $h_\alpha'(0, r_1, r_2, \cdots)$, with total dimensions $t-s$ equal to $\dim y_\alpha + \sum 2r_i(p^i - 1) - 2$.

If the integers $\dim y_\alpha$ are all even, then everything in the spectral sequence is even dimensional. It follows that $\{X, \boldsymbol{Y}\}_m$ is zero for m odd. Together with Lemma 4, this completes the proof of Theorem 1.

3. Computation of $H^*(B(U_n); Z_p)$ and $H^*(\boldsymbol{M}(U); Z_p)$. This section will complete the study of $M(U_n)$ by constructing a stable object

$$\boldsymbol{M}(U) = (o, o, M(U_1), SM(U_1), M(U_2), SM(U_2), \cdots);$$

and showing that $H^*(\boldsymbol{M}(U); Z_p)$ is a free module over $A/(Q_0)$, with even dimensional generators, for any prime p.

The proof of this assertion is an immediate generalization of the argument which Thom used to compute the non-orientable cobordism group. In our terminology, Thom showed that $H^*(\boldsymbol{M}(O); Z_2)$ is a free A-module. (See [15, pp. 39-42].)

First a description of $H^*B(U_n)$. *The coefficient group Z_p is to be*

understood, where p is some fixed prime. (However, integer coefficients could equally well be used.) Let $T_n \subset U_n$ be the n-torus consisting of diagonal unitary matrices. There is a natural map $B(T_n) \to B(U_n)$ of classifying spaces. The cohomology algebra $H^*B(T_n)$ is a polynomial algebra on generators $t_1, \cdots, t_n$ of dimension 2. According to Borel and Serre [4] we may identify $H^*B(U_n)$ with the subalgebra consisting of all symmetric polynomials.

A basis for $H^{2r}B(U_n)$ over Z_p is given as follows. Let $\omega = i_1 \cdots i_k$ range over all partitions of r such that the "length" k is less than or equal to n. (A partition of r is an unordered sequence of positive integers with sum r.) *Define* $s(\omega)$ as the "smallest" symmetric polynomial which contains the term $t_1^{i_1} \cdots t_k^{i_k}$.

[The notation $\sum t_1^{i_1} \cdots t_k^{i_k}$ is commonly used. A more precise definition would be the following. Consider all distinct monomials which can be obtained from $t_1^{i_1} \cdots t_k^{i_k}$ by permuting the n variables; and let $s(\omega)$ denote their sum. It is clear that these elements $s(\omega)$ from a basis for the vector space of symmetric polynomials.]

Next we must study the Thom complex $M(U_n)$. For a group $G \subset SO_m$ recall that $M(G)$ is the quotient space $E/\partial E$, where E is an oriented m-disk bundle over $B(G)$. Any CW-cell subdivision of $B(G)$ induces a cell subdivision of $M(G)$ as follows. For each open i-cell e of $B(G)$, the inverse image e' in $E - \partial E$ is an $(i+m)$-cell. Clearly, $M(G)$ is the disjoint union of these cells e', together with the base point. It is not difficult to verify that $M(G)$ thus becomes a CW-complex.

Let $G \times 1$ denote the group G, considered as a subgroup of SO_{m+1}. The CW-complex $M(G \times 1)$ can be identified with the suspension $SM(G)$ as follows. Let D^m denote the m-disk and I the unit interval. Map $D^m \times I$ onto D^{m+1} by the correspondence

$$(x_1, \cdots, x_m), y \to x_1, \cdots, x_m, (2y-1)(1 - x_1^2 - \cdots - x_m^2)^{\frac{1}{2}}.$$

This correspondence gives rise to a map f of $E \times I$ onto the total space E_1 of the associated $(m+1)$-disk bundle. Since f carries $(\partial E \times I) \cup (E \times \partial I)$ onto the boundary ∂E_1, it follows that f gives rise to a map $f'\colon SM(G) \to M(G \times 1)$. But f is a relative homeomorphism, hence f' is a homeomorphism.

The Thom isomorphism

$$\phi\colon H^iB(G) \to H^{i+m}(M(G) \bmod o)$$

is defined as follows. (see [14, Théorème I.4]). The cohomology of $M(G) \bmod o$ will be identified with the cohomology of $E \bmod \partial E$. It can be

verified that $H^m(E \bmod \partial E; Z)$ is an infinite cyclic group, with standard generator u. The isomorphism ϕ is now defined by the formula $\phi(a) = \pi^*(a)u$, where $\pi\colon E \to B(G)$ denotes the projection map. It follows from this definition that the following diagram is commutative:

$$\begin{array}{ccc} H^{i+m}(M(G) \bmod o) & \xrightarrow{S} & H^{i+m+1}(M(G \times 1) \bmod o) \\ \uparrow \phi & & \uparrow \phi \\ H^iB(G) & = & H^iB(G \times 1). \end{array}$$

Here S denotes the cohomology suspension, defined using the cohomology cross product.

Now let us specialize to the case $G = U_n \subset SO_{2n}$. The classifying space $B(U_n)$ has a standard cell subdivision due to Ehresmann [7] and $B(U_n)$ is a subcomplex of $B(U_{n+1})$. Hence $M(U_n)$ is a CW-complex and the two-fold suspension

$$S^2M(U_n) = M(U_n \times 1 \times 1)$$

is a subcomplex of $M(U_{n+1})$. Thus

$$\boldsymbol{M}(U) = (0, 0, M(U_1), SM(U_1), M(U_2), \cdots)$$

is a stable object. The track group $\{S^0, \boldsymbol{M}(U)\}_k$ is clearly isomorphic to the stable homotopy group $\pi_{k+2n}(M(U_n))$, with n large.

On the other hand the complexes $B(U_1) \subset B(U_2) \subset \cdots$ have a union $B(U)$ which is again a CW-complex. The isomorphisms

$$\phi\colon H^iB(U_n) \to H^{i+2n}(M(U_n) \bmod o)$$

give rise, in the limit, to an isomorphism

$$\phi\colon H^iB(U) \to H^i\boldsymbol{M}(U).$$

It follows that $H^*\boldsymbol{M}(U)$ has a basis over Z_p consisting of the elements $\phi s(\omega)$, where ω ranges over all partitions.

THEOREM 2. *The cohomology $H^*\boldsymbol{M}(U)$ with coefficient group Z_p is a free module over $A/(Q_0)$, having as basis the elements $\phi s(\lambda)$, where λ ranges over all partitions which contain no integer of the form $p^j - 1$.*

Together with Theorem 1, and the fact that $\boldsymbol{M}(U)$ has no odd dimensional cohomology, this clearly implies the following.

THEOREM 3. *The groups $\{S^0, \boldsymbol{M}(U)\}_m$ have no torsion.*

The full structure of these stable homotopy groups can now be determined, using the fact that the stable Hurewicz homomorphism

$$\{S^0, \boldsymbol{Y}\}_m \rightarrow H_m(\boldsymbol{Y}; Z)$$

is a $\mathcal{C}$-isomorphism, where $\mathcal{C}$ denotes the class of finite groups. (See Serre [12] for definitions. This particular assertion is not in Serre's paper, but is well known.)

COROLLARY. *The group* $\{S^0, \boldsymbol{M}(U)\}_m = \pi_m \boldsymbol{M}(U)$ *is zero for* m *odd, and is free abelian for* $m = 2n$, *the number of generators being equal to the number of partitions of* n.

The proof of Theorem 2 will be based on a peculiar partial ordering of partitions, due to Thom. Given a sequence $R = (r_1, r_2, \cdots)$, define ω_R as the partition of $\sum r_j(p^j - 1)$ consisting of r_j copies of $p^j - 1$ for each $j \geqq 1$. Thus every partition ω can be written uniquely in the form $\lambda\omega_R$, where $\lambda = h_1 \cdots h_l$ contains no integer of the form $p^j - 1$. Let l denote the length of λ and let $\Sigma = h_1 + \cdots + h_l$ denote the sum of the integers in λ. Similarly, given a second partition ω', define l' and Σ'.

Definition. ω' *is less than* ω if $l' < l$, or if $l' = l$ and $\Sigma' > \Sigma$. (Note that integers of the form $p^j - 1$ are completely ignored in this definition.)

LEMMA 6. *The cohomology operation* $\mathfrak{P}^R$ *carries* $\phi s(\lambda) \in H^*\boldsymbol{M}(U)$ *into* $\phi s(\lambda\omega_R)$ *plus a linear combination of elements* $\phi s(\omega')$ *with* ω' *less than* $\lambda\omega_R$.

Proof. It is clearly sufficient to prove the corresponding assertion for $H^*M(U_n)$, where n is large (say $n \geqq l + r_1 + r_2 + \cdots$), but finite. Consider the cross-section

$$f \colon B(U_n) \rightarrow E, \partial E$$

of the $2n$-disk bundle, determined by the center points of the disks. The induced cohomology homomorphism f^* carries the fundamental cohomology class $u \in H^{2n}(E \bmod \partial E)$ into the characteristic class

$$c_n = t_1 \cdots t_n = s(1 \cdots 1) \in H^{2n}B(U_n).$$

(See Thom [14], Borel and Serre [4].) Hence f^* carries the general element $\phi(a) = \pi^*(a)u \in H^{i+2n}(E \bmod \partial E)$ into the cup product $ac_n \in H^{i+2n}B(U_n)$. But the correspondence $a \rightarrow ac_n$ is a monomorphism; hence f^* is a monomorphism. Thus in order to prove Lemma 6 it is sufficient to prove the following.

Assertion. *$\mathcal{P}^R(s(\lambda)c_n)$ is equal to $s(\lambda\omega_R)c_n$ plus a linear combination of elements $s(\omega')c_n$ with ω' less than $\lambda\omega_R$.*

Consider a typical monomial $t_1^{a_1}\cdots t_n^{a_n}$ of the sum $s(\lambda)c_n$. Here l of the integers $a_1,\cdots,a_n$ are equal to the integers $1+h_1,\cdots,1+h_l$ in some order; while the remaining $n-l$ integers a_i are equal to 1. According to Lemma 3 we have

$$\mathcal{P}^R(t_1^{a_1}\cdots t_n^{a_n}) = \sum_{R_1+\cdots+R_n=R} (\mathcal{P}^{R_1}t_1^{a_1})\cdots(\mathcal{P}^{R_n}t_n^{a_n}).$$

This formula is valid even for the case $p=2$, since $B(U_n)$ has no odd dimensional cohomology. (See Lemma 3′.) The expression $\mathcal{P}^{R_i}t_i^{a_i}$ is equal to some constant k_i times $t_i^{b_i}$, where $b_i \geqq a_i$. The case $b_i = a_i$ can occur only if $R_i = 0$.

Each such monomial $(k_1\cdots k_n)t_1^{b_1}\cdots t_n^{b_n}$ contributes to a symmetric polynomials $s(\omega')c_n$, where ω' denotes the partition obtained from the sequence $b_1-1,\cdots,b_n-1$ by deleting zero. We wish to choose $R_1,\cdots,R_n$ so that this partition ω' is as "large" as possible, in the sense of the partial ordering. The first requirement is that as few as possible of the integers b_i-1 should be of the form p^j-1. But if $a_i=1$, and if the constant k_i is non-zero, then $\mathcal{P}^{R_i}t_i^{a_i}$ is necessarily of the form $t_i^{p^j}$. (See Lemma 3.) Thus the best we can do is to *choose $R_1,\cdots,R_n$ so that b_i is a power of p only if $a_i=1$.*

The second requirement in order to make ω' "large" is that the sum of all b_i-1 for which b_i is not a power of p should be as small as possible. Evidently, the best we can do in this direction is to *choose $R_i=0$ whenever $a_i>1$*; so that b_i will be equal to a_i whenver $a_i>1$.

Now consider the sum of all terms $(\mathcal{P}^{R_1}t_1^{a_1})\cdots(\mathcal{P}^{R_n}t_n^{a_n})$ for which this last condition (that R_i must be equal to zero whenever $a_i>1$) is satisfied. Each such term has the form $t_1^{b_1}\cdots t_n^{b_n}$, where l of the integers $b_1,\cdots,b_n$ are equal to $1+h_1,\cdots,1+h_l$ in some permutation; and the remaining $n-l$ integers b_i are powers of p. Recall that $\mathcal{P}^{R_i}t_i$ is equal to $t_i^{p^j}$ if $R_i=\Delta_j$ and is zero otherwise. Hence the relation $R_1+\cdots+R_n=R=(r_1,r_2,\cdots)$ implies that a given power p^j, $j\geqq 1$, must occur exactly r_j times in the sequence $b_1,\cdots,b_n$. The integer 1 must therefore occur $n-l-r_1-r_2-\cdots$ times in the sequence $b_1,\cdots,b_n$. Taking the sum of all monomials $t_1^{b_1}\cdots t_n^{b_n}$ which satisfy these conditions, we obtain exactly the polynomial $s(\lambda\omega_R)c_n$. This completes the proof of Lemma 6.

Proof of Theorem 2. The equations

$$\mathcal{P}^R\phi s(\lambda) = \phi s(\lambda\omega_R) + \sum(\text{constant})\phi s(\lambda'\omega_{R'}),$$

with all λ' less than λ, can be solved inductively, giving rise to equations:

$$\phi s(\lambda\omega_R) = \mathcal{P}^R\phi s(\lambda) + \sum(\text{constant})\mathcal{P}^{R'}\phi s(\lambda'),$$

with all λ' less than λ. (Only a finite number of terms are involved, since $H^*\boldsymbol{M}(U)$ is finitely generated in each dimension.) But the elements $\phi s(\lambda\omega_R)$ are known to form a Z_p-basis for $H^*\boldsymbol{M}(U)$. Therefore the elements $\mathcal{P}^R\phi s(\lambda)$ also form a Z_p-basis for $H^*\boldsymbol{M}(U)$. Since $\{\mathcal{P}^R\}$ is a basis for the vector space $A/(Q_0)$ over Z_p, this implies that the elements $\phi s(\lambda)$ form an $A/(Q_0)$-basis for $H^*\boldsymbol{M}(U)$. This completes the proof of Theorem 2, and hence Theorem 3.

4. Cohomology computations for $\boldsymbol{B(SO_{2n})}$ and $\boldsymbol{M(SO)}$. Consider the torus $T_n \subset U_n \subset SO_{2n}$, and the corresponding homomorphism

$$H^*(B(SO_{2n});Z_p) \to H^*(B(T_n);Z_p).$$

According to Borel and Serre [4], if p is odd, then the first algebra may be identified with the subalgebra of the second consisting of all polynomials $a + t_1 \cdots t_n b$, where a and b are symmetric polynomials in the elements $t_1^2, \cdots, t_n^2$. Thus a basis for $H^*(B(SO_{2n});Z_p)$ over Z_p is given by the elements $s(\omega)$ and $s(\omega)t_1 \cdots t_n$, where $\omega = i_1 \cdots i_k$, $k \leqq n$, is a partition into even integers. Letting n tend to infinity, *a Z_p-basis for $H^*(B(SO);Z_p)$ is given by the elements $s(\omega)$, where ω ranges over all partitions into even integers.*

Carrying out an argument completely analogous to that in Section 3, we construct a stable object

$$\boldsymbol{M}(SO) = (o, M(SO_1), M(SO_2), \cdots),$$

and prove the following.

THEOREM 4. *Let p be an odd prime, and let $\lambda = h_1 \cdots h_l$ range over all partitions into integers h_i which are even and not of the form $p^j - 1$. Then $H^*(\boldsymbol{M}(SO);Z_p)$ is the free $A/(Q_0)$-module having as basis the elements $\phi s(\lambda)$.*

Together with Theorem 1 this proves the following

THEOREM 5. *The cobordism groups $\Omega^i = \pi_i(\boldsymbol{M}(SO))$ contain no odd torsion.*

C. T. C. Wall has recently proved that an element in the 2-torsion subgroup of Ω^i is completely determined by its Stiefel-Whitney numbers. Together with Theorem 5, this proves the following conjecture of Thom.

Corollary 1. *If the Stiefel-Whitney numbers and the Pontrjagin numbers of a compact, oriented, differentiable manifold V^i are all zero, then V^i is a boundary.*

As special cases:

Corollary 2. *Suppose that V^i can be imbedded in euclidean space so as to have trivial normal bundle. Then V^i is a boundary.*

The proof is clear.

Corollary 3. *Suppose that $H_*(V^i;Z_2)$ is isomorphic to $H_*(S^i;Z_2)$. Then V^i is a boundary.*

Proof. The Stiefel-Whitney number $w_i[V^i]$ is equal to the Euler characteristic reduced modulo 2; hence is zero. If $i=4n$, then the Pontrjagin number $p_n[V^i]$ is zero by the index theorem (Hirzebruch [8]). Since the other characteristic numbers of V^i are trivially zero, it follows that V^i is a boundary.

Concluding Remarks. There are other homotopy groups which may be accessible, using the Adams spectral sequence. For example, the symplectic groups $Sp(n)\subset SO_{4n}$ give rise to a stable object

$$\boldsymbol{M}(Sp)=(o,o,o,o,M(Sp(1)),SM(Sp(1)),$$
$$S^2M(Sp(1)),S^3M(Sp(1)),M(Sp(2)),\cdots).$$

Assertion. The groups $\pi_i\boldsymbol{M}(Sp)$ have no odd torsion.

This can be proved directly from the spectral sequence; or can be derived from Theorem 5, using the natural map $\boldsymbol{M}(Sp)\to\boldsymbol{M}(SO)$.

Problem. Can one compute the spectral sequence for $\pi_*\boldsymbol{M}(Sp)$ corresponding to the prime $p=2$?

Similarly, the representations $Spin(n)\to SO_n$ give rise to a stable object.

$$\boldsymbol{M}(Spin)=(o,M(Spin(1)),M(Spin(2)),\cdots).$$

Again there is no odd torsion; but the case $p=2$ seems difficult. As a final question, consider the stable object $\boldsymbol{M}(SU)$ corresponding to the special unitary group.

Problem. What can be said about $\pi_*\boldsymbol{M}(SU)$?

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SOLUTION OF SOME PROBLEMS OF DIVISION.*

Part IV. Invertible and Elliptic Operators.[1]

By L. Ehrenpreis.

1. Introduction. Let V be a topological vector space and $L: V \to V$ a continuous linear map. The study of the linear equation

$$Lv = w \qquad (v, w \in V) \tag{1}$$

leads us to three very natural questions:

Problem A. What is the image of L, that is, for what w does (1) have a solution?

Problem B. Let w have some additional properties, that is, let w belong to some subspace W of V; what additional properties does v have?

Problem C. What is the lack of uniqueness of (1)? That is, describe completely the kernel of L, the set of v which satisfy $Lv = 0$.

In this series we study mainly the following case: V is a space of distributions or functions, and L is a convolution $Lv = a * v$, where a is a certain distribution such that $v \to a * v$ is a continuous map of V into V. For many spaces V, which can be describel roughly by saying that the Fourier transform of the dual V' of V is a space of entire functions which is described by growth conditions at infinity, these growth conditions depending only on the distance from the origin, it was shown in Part III (see [7]) that L is always onto. On the other hand, for many interesting spaces, e. g., the space $\mathcal{E}$ of indefinitely differentiable functions on $R = R^n =$ Euclidean n space or the $\mathcal{D}'$ of distributions on R (see [24]) there exist continuous convolution maps which are not onto (an example is convolution by an indefinitely differentiable function of compact carrier). Thus, problem A takes on a non-trivial form for these spaces. We shall give first a partial solution to problem A by solving completely

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Problem A′. Determine those L for which L' is onto.

Definition. If $L\colon V \to V$ is onto, we call L *invertible.* If $Lv = \alpha * v$, we say α is invertible if L is.

Thus, we shall determine all invertible elements of $\mathcal{E}'$ considered as convolution operators of $\mathcal{D}'$ into $\mathcal{D}'$, or $\mathcal{E}$ into $\mathcal{E}'$.

We denote by $\boldsymbol{E}'$ the Fourier transform of $\mathcal{E}'$; thus $\boldsymbol{E}'$ is the space of all entire functions of exponential type on $C = C^n$ which are of polynomial increase on $R = R^n$ which is the real part of C (see [24], [11]).

Definition. A function $J \in \boldsymbol{E}'$ is called *slowly decreasing* if there exists a positive number a such that for each point $x \in R$ we can find a point $y \in R$ with

$$|x - y| \leqq a \log(1 + |x|) \tag{2}$$

$$|J(y)| \geqq (a + |y|)^{-a}. \tag{3}$$

Then our first main result is

THEOREM I. *$S \in \mathcal{E}'$ is invertible for $\mathcal{D}'$ (or for $\mathcal{E}$) if and only if the Fourier transform of S is slowly decreasing. In order for S to be invertible it is sufficient to be able to solve the equation $S * T = S'$ for $T \in D'$, where S' is a fixed invertible distribution in E'.*

Since $\delta =$ Dirac's measure is clearly invertible, Theorem I contains as a special case a conjecture of L. Schwartz (see [24]).[2]

Theorem I was announced in a Proceedings note [9].

In case S is not invertible, we could, of course, again ask the question (Problem A) as to what is $S * \mathcal{D}'$ of $S * \mathcal{E}$. In Part II (see [7] p. 692) we stated the conjecture that if $S \in \mathcal{D}$ then $S * \mathcal{D}' \neq \mathcal{E}$. We shall show that, in fact, much more is true (see Theorem 2.5 below): If S is not invertible, then it is not even true that $\mathcal{D} \subset S * \mathcal{D}'$.

We can also give a more complete answer to Problem A of determining the image $S * \mathcal{D}'$ (or $S * \mathcal{E}$) for any $S \in \mathcal{E}'$. A trivial necessary and sufficient condition for $T \in \mathcal{D}'$ to be in $S * \mathcal{D}'$ is that, if $f_\alpha \in \mathcal{D}$, $S * f_\alpha \to 0$ in $\mathcal{D}$, then $T \cdot f_\alpha \to 0$, that is, T is continuous on the topology τ defined on $\mathcal{D}$ as follows: N is a neighborhood of zero in τ if we can find a neighborhood N' of zero in $\mathcal{D}$ so that N consists of all $f \in \mathcal{D}$ with $S * f \in N'$. Now, a good description

[2] It is incorrectly stated by Schwartz in [24] that the result for $n = 1$ can be proved by the methods of the theory of mean periodic functions. Professor Schwartz has kindly pointed out to me that his proof only shows that if $S * T = \delta$ has a solution $T \in \mathcal{D}'_F$, then $S * \mathcal{D}'_F = \mathcal{D}'_F$ (see below). This result was extended to $n > 1$ by Malgrange [21].

of $S * \mathfrak{D}'$ would be obtained if we give a "good" description of τ. This seems to be extremely difficult and we shall content ourselves with giving a partial solution to this problem (Section 2).

In addition, we shall give in Section 2 several other necessary and sufficient conditions for invertibility of S. One such is (see Theorem 2.6 below): For any entire function G, $JG \in \boldsymbol{D}$ implies $G \in \boldsymbol{D}$. (Here J is the Fourier transform of S and $\boldsymbol{D}$ is the Fourier transform of the space D of L. Schwartz [24].)

We shall prove the analog of Theorem I for the space $\mathfrak{D}'_F$ of distributions of finite order.

It might be suspected that for $S \in \mathcal{E}'$, even if S is not invertible for $\mathfrak{D}'$, we should be able to find a larger space $\mathfrak{D}'_M$ which is the dual of a space of Carleman non quasi-analytic functions (see e.g. [14]) such that for some $T \in \mathfrak{D}'_M$ we can solve the equation $S * T = \delta$. However, I shall produce an S for which there cannot exist a $T \in \mathfrak{D}'_M$ for any Carlemann non quasi-analytic class (see Theorem 6.2 below). I should only remark that the existence of this example came as a very great surprise to me personally.

Section 3 is devoted towards proving that in case S is invertible (for $\mathfrak{D}'$ or $\mathcal{E}'$) then L. Schwartz' mean periodic expansion (see [27]) for a solution f of $S * f = 0$ in terms of the exponential polynomial solutions can be greatly simplified.

The next question which we shall discuss (Section 4) is: Let $T \in \mathfrak{D}'$ have the property that for any $S \in \mathcal{E}'$ the equation $S * U = T$ has a solution $U \in \mathfrak{D}'$; what can be said about T? That is what is the intersection of all $S * \mathfrak{D}'$ for $S \in \mathcal{E}'$? We shall prove

THEOREM II. *For $T \in D'$, a necessary and sufficient condition that for each $S \in \mathcal{E}'$ the equation $S * U = T$ has a solution $U \in \mathfrak{D}'$ is that T be real analytic. In fact,* $\bigcap_{S \in \mathcal{E}'} S * \mathfrak{D}' = \bigcap_{S \in \mathcal{E}'} S * \mathcal{E} =$ *real analytic functions. In case T is analytic in a strip in S around R, then for any $S \in \mathcal{E}'$ there exists a U which is analytic in a strip in C around R such that $S * U = T$.*

The proof of Thetorem II depends on the Denjoy-Carleman Theorem for quasi-analytic functions (see [22]).

In Section 5 we discuss a special case of Problem B: We say that the distribution T is C^∞ in x_1 if we can find a sequence of positive numbers a_j so that $\{a_j(\partial^j/\partial x_1{}^j)T\}$ is a bounded set in $\mathfrak{D}'$, or, what can be seen to be the same thing, that T belongs to the topological tensor product (see [19]) of $\mathcal{E}(x_1)$ with $\mathfrak{D}'(x_2, \cdots, x_n)$. (A proof of this assertion can be obtained by

use of methods of [12].) Then we want to find all $S \in \mathcal{E}'$ so that if $S * U$ is C^∞ in x_1, then U is C^∞ in x_1. If S has this property, then we say that S is C^∞ elliptic in x_1. Our main result for this problem is

THEOREM III. *Let $S \in \mathcal{E}'$ be invertible for $\mathcal{D}'$. S is C^∞ elliptic in x_1 if and only if for each $r \geqq 0$ we can find a $b_r > 0$ with the property that*

$$(1.1) \qquad |\, \mathfrak{I} z \,| \geqq r \log(1 + |\, z_1 \,|)$$

whenever $z \in V$ and

$$(1.2) \qquad |\, z_1{}^r \,| \geqq b_r(1 + |\, z \,|).$$

It should be remarked that conditions (1.1) and (1.2) could be condensed to the single condition:

$$(1.3) \qquad |\, z_1 \,|^r \exp(-d \,|\, \mathfrak{I} z \,|) \leqq d_r(1 + |\, z \,|)^d.$$

A similar remark applies to all the other cases of ellipticity discussed below.

We also show by an example (Example 4 of Section 5 below) that if S is not invertible for $\mathcal{D}'$, then conditions (1.1) and (1.2) do not suffice even to guarantee that all distribution solutions of $S * T = 0$ be C^∞ in x_1.

We call the distribution T *entire* in x_1 if for any $a > 0$ the set $\{(a^j/j!)(\partial^j T/\partial x_1{}^j)\}$ is bounded in $\mathcal{D}'$ or, what is the same thing, if T belongs to the topological tensor product of the space of entire functions in x_1 with the space $\mathcal{D}'(x_2, \cdots, x_n)$. We say that $S \in E'$ is entire elliptic in x_1 if whenever $S * T$ is entire in x_1, then T is entire in x_1. We prove the analog of Theorem III for entire ellipticity.

We could define similarly classes between C^∞ ellipticity and entire ellipticity, but we shall not do this since the methods of the present paper apply without essential modifications.

We prove also that if S is C^∞ elliptic in all variables, then it is necessarily invertible.

In the above we have used the space $\mathcal{D}'$ to define ellipticity. We could make a similar definition for the space $\mathcal{D}'_F$ (see [5]) of distributions of finite order; we call this ellipticity *weak* ellipticity (thus, weak C^∞ elliptic; weak elliptic, etc.). Now there is no difference between weak entire ellipticity and entire ellipticity, and if S is a differential operator, then weak C^∞ ellipticity and C^∞ ellipticity are the same. However, if $S \cdot f = df(0)/dx + f(1)$, then S is weakly C^∞ elliptic but S is not C^∞ elliptic.

Finally, we show certain special properties of elliptic operators. For example, if S is entire elliptic in x_1, then, in x_1, S is the composition of a

translation with a differentiation. Thus, if S is entire elliptic in all variables, then it is the composition of a classical elliptic differental operator with a translation. If S is a differential difference operator in x_1 which is C^∞ elliptic in x_1, then in x_1, S is the composition of a translation with a differentiation. Thus, if S is a differential-difference operator in all variables which is C^∞ elliptic in all variables, then S is the composition of a translation with a partial differential operator which is C^∞ elliptic in all variables. The above example $S\cdot f = df(0)/dx + f(1)$ shows that the analog of the last proposition is false for weak C^∞ ellipticity.

Suppose that $T \in \mathfrak{D}'$ is C^∞ in $x_1, \cdots, x_r$ $(r < n)$. Then we prove easily (see Proposition 5.1) that for h an indefinitely differentiable of compact support in $x_{r+1}, \cdots, x_n$ we have $T * h \in \mathcal{E}$. (Here $T * h$ is the convolution of T with the direct product of h with $\delta(x_1, \cdots, x_r)$.) However, the converse is not true, e. g., for such an h we have $\delta(x_1 - x_2) * h(x_2) = h(x_1 - x_2) \in \mathcal{E}$, while $\delta(x_1 - x_2)$ is not C^∞ in x_1. If T has the property that $T * h \in \mathcal{E}$ for all such h, then we can prove easily by means of the closed graph theorem that $h \to (T * h)(0)$ is a distribution on these h which we call the restriction of T to the plane $x_1 = x_2 = \cdots = x_r = 0$.

This leads to the following concept: Let us partition the variables $x = (x_1, x_2, \cdots, x_n)$ into three sets, say $x = (x', x'', x''')$. Then we say that $T \in \mathfrak{D}'$ is C^∞ *in x' relative to x'''* if for any $h \in \mathfrak{D}(x''')$, we have $T * h$ is C^∞ in the variables x'. We say that $S \in \mathcal{E}'$ is C^∞ *elliptic in x' relative to x'''* if whenever $T \in \mathfrak{D}'$ is C^∞ in x' relative to x''' and $W \in \mathfrak{D}'$ satisfies $S * W = T$, then W is also C^∞ in x' relative to x'''. For simplicity of notation shall consider only the case when the variables x'' are absent, but there is no difficulty in extending all our results to the general case.

In Section 5 we describe all invertible $S \in \mathcal{E}'$ which are C^∞ elliptic in x' relative to x'''. In particular, if S is a differential operator in x_1 with leading coefficient 1, then S is C^∞ elliptic in x_1 relative to $(x_2, \cdots, x_n)$. Thus, a distribution solution of any linear constant coefficient partial differential equation has a restriction to any non-characteristic hypersurface.

We can define also similar concepts for entire ellipticity and weak ellipticity. The classes of $S \in \mathcal{E}'$ which satisfy these conditions are described in Section 5.

The results of previous authors on the problem of ellipticity deal with the case where S is a linear partial differential operator, and the only kind of ellipticity considered is simultaneous ellipticity in all variables.[3] In this

[3] I have been just informed by L. Gärding that he and B. Malgrange have characterized all linear constant coefficient partial differential operators which are C^∞ elliptic in $(x_1, \cdots, x_r)$ relative to $(x_{r+1}, \cdots, x_n)$.

case, the analog of Theorem III for entire ellipticity was found by Petrowski [27], while the analog for C^∞ ellipticity was obtained by Hörmander [20]. We should like to mention that the results of Theorem III were announced in the Proceedings of the National Academy of Sciences [10] and they were obtained independently of those of Hörmander (though Hörmander's results appeared slightly before mine).

In case S is a linear constant coefficient partial differential operator, then a complete solution to Problem C will be given in a future publication (see [15], [16]).

The problem of extending the results of this paper to continuous linear transformations $L\colon \mathcal{D}' \to \mathcal{D}'$ which are not of the form $L(T) = S * T$ for some $S \in \mathcal{E}'$ is undoubtedly very difficult.

The paper concludes with a list of unsolved problems and some general remarks.

The notations in this paper will be the same as those used in Parts I, II, and III.

I should like to thank my friend Dr. D. J. Phlotzelphlip for several useful discussions. I should also like to thank Professor Beurling from whom I have learned much.

Added in proof. Since this paper was written, some of the questions posed have been answered. I have inserted an appendix at the end to give some indications of the progress.

2. Invertible operators for $\mathcal{D}'$, $\mathcal{E}'$, $\mathcal{D}'_F$. In this section we shall first find all invertible operators for $\mathcal{D}'$. In a previous paper [7] we proved that all differential-difference operators were invertible for $\mathcal{D}'$. The proof consisted essentially of two parts: (a) Describe explicitly the topology of the Fourier transform $\boldsymbol{D}'$ of $\mathcal{D}$. (b) Use the fact that exponential polynomials (the Fourier transform of differential-difference operators) do not tend to zero too fast at infinity "too often," and use the minimum modulus theorem to take care of the points where the exponential polynomials are small. We want to give a simplified abstract treatment of (a) and (b).

Let then A be a topological vector space of entire functions on C. We assume that the topology of A can be described as follows: There exist continuous positive functions $\{H\}$ on C so that a fundamental system of neighborhoods of zero in A consists of the sets N_H comprising all functions $f \in A$ such that $|f(z)| \leqq H(z)$ for all $z \in C$.

This is part (a) described above; now to part (b): Let $z_0 \in C$ and let α

be a subset of C; we say that α *surrounds* z_0 if for every entire function g

$$|g(z_0)| \leqq \max_{z \in \alpha} |g(z)|. \tag{2.1}$$

Now, let J be an entire function for which $f \to Jf$ defines a continuous map of $A \to A$. We want to know when $Jf \to f$ is a continuous map of $JA \to A$. (JA is the space of Jf, $f \in A$ with the topology induced from A.) Suppose each point $z_0 \in C$ can be surrounded by a set α on which J is large, say $|J(z)| \geqq b(z_0)$ for all $z \in \alpha$. Then how large does $b(z_0)$ have to be in order to guarantee that $Jf \to f$ is continuous? Let H be one of the functions above used to define the topology of A; when can we find another such function H' so that the conditions $|J(z)f(z)| \leqq H'(z)$ should imply $|f(z)| \leqq H(z)$? We know from the above that if $|J(z_0)f(z_0)| \leqq H'(z_0)$, then on α we have $|f(z)| \leqq H'(z)[b(z_0)]^{-1}$. Thus, since α surrounds z_0 and f is entire, it follows that

$$|f(z_0)| \leqq [b(z_0)]^{-1} \max_{z \in \alpha} H'(z). \tag{2.2}$$

Putting the above together we have

LEMMA 2.1. *With the above notation, suppose for every H we can find an H' so that for all $z_0 \in C$,*

$$[b(z_0)]^{-1} \max_{z \in \alpha} H'(z) \leqq H(z_0), \tag{2.3}$$

then $Jf \to f$ is a continuous linear map of $JA \to A$.

Finally, there remains the problem of constructing the sets α. We shall construct them by means of the minimum modulus theorem. We suppose now that J is an entire function of exponential type B which is bounded on R (the case of an arbitrary $J \in \mathbf{E}'$ is easily reduced to this). Suppose that for each $x \in R$ we can find a $y \in R$ so that $|x - y| \leqq p(x)$ and $J(y) \geqq q(x)$. Let $y_2, \cdots, y_n$ be fixed and draw about the point y_1 in the complex plane a circle $\beta(y_2, \cdots, y_n)$ of center y_1 and radius between $p(x)$ and $2p(x)$ on which

$$|J(w_1, y_2, \cdots, y_n)| \geqq M\, |J(y_1, y_2, \cdots, y_n)|^d \exp(-dBp(x)). \tag{2.3}$$

Here d is a positive integer and M depends only on J. The existence of such a circle is guaranteed by the minimum modulus theorem (see [8]). Moreover, it is clear that if $x_2 = y_2, \cdots, x_n = y_n$, then x is surrounded by this circle.

If it is not true that x is surrounded by β, then we just iterate the above process. For $w_1 \in \beta, y_3, \cdots, y_n$ fixed, we can draw in the complex plane a

circle $\beta(w_1, y_3, \cdots, y_n)$ of center y_2 and radius between $p(x)$ and $2p(x)$ on which

$$(2.4) \qquad |J(w_1, w_2, y_3, \cdots, y_n)| \geqq M_1 \, |J(y_1, y_2, \cdots, y_n)|^d \exp(-2dBp(x)).$$

The existence of such a circle is again guaranteed by the minimum modulus theorem. Suppose for simplicity that $y_3 = x_3, y_4 = x_4, \cdots, y_n = x_n$. Then I claim we could choose

$$(2.5) \qquad \alpha = \{(w_1, w_2, y_3, \cdots, y_n)\}_{w_2 \in \beta(w_1, y_3, \cdots, y_n)}.$$

For, given any entire function $g(w)$, we have for any $w_1 \in \beta(y_2, y_3, \cdots, y_n)$,

$$(2.6) \qquad |g(w_1, x_2, y_3, \cdots, y_n)| \leqq \max_{w_2 \in \beta(w_1, w_2, y_3, \cdots, y_n)} |g(w_1, w_2, y_3, \cdots, y_n)|$$

by the maximum modulus theorem. Hence, again by the maximum modulus theorem, we have

$$(2.7) \qquad |g(x_1, x_2, y_3, \cdots, y_n)| \leqq \max_{w_1 \in \beta(y_2, y_3, \cdots, y_n)} |g(w_1, x_2, y_3, \cdots, y_n)|.$$

Thus, our assertion is proven.

Finally, if it is not true that $y_3 = x_3, y_4 = x_4, \cdots, y_n = y_n$, we just continue the above process.

In case $y \in R$ but $x \notin R$, the same considerations apply. We have thus proven

LEMMA 2.2. *In the above notation, given any $z_0 \in R$, we can surround z_0 by a set α on which*

$$(2.8) \qquad |J(z)| \geqq M[q(x)]^d \exp(-dBp(x)),$$

where d and M are constants depending only on J. Morevoer, we have

$$(2.9) \qquad \max_{z \in \alpha} |z - z_0| \leqq Mp(z_0).$$

Now we are ready to apply the above to the space $\boldsymbol{D}$. For this purpose, we have to describe the topology of $\boldsymbol{D}$ in a manner similar to the space A above. This is

THEOREM 2.1. *Let $H(z)$ be any continuous positive function on C such that if h is any continuous function on C for which we can find an $a > 0$ so that*

$$(2.10) \qquad h(z) = O(\exp(a\,|\mathfrak{J}(z)|)/(1 + |z|^m)$$

for all m, then also $h(z) = O(H(z))$. Call N_H the set of $F \in \boldsymbol{D}$ for which $|F(z)| \leqq H(z)$ for all $z \in C$. Then the sets N_H form a fundamental system of neighborhoods of zero in $\boldsymbol{D}$.

Proof. First we show that the sets N_H are neighborhoods of zero in $\boldsymbol{D}$. Now, each N_H is convex and clearly for any $F \in \boldsymbol{D}$ we can find a $b > 0$ so that $bF \in N_H$. Let B be a bounded set in $\boldsymbol{D}$; set

$$h(z) = \max_{F \in B} |F(z)| \tag{2.11}$$

for all $z \in C$. Then, by the explicit description of the bounded sets of $\boldsymbol{D}$ (see [5], [11]), we see that h satisfies (2.10). Thus, we can find a $b' > 0$ so that $b'B \subset N_H$. Since $\boldsymbol{D}$ is bornologic (see [4]), it follows that N_H is a neighborhood of zero in $\boldsymbol{D}$.

We now have to show that the neighborhoods N_H are fundamental in $\boldsymbol{D}$. But this is an immediate consequence of Theorem 1 of [7] which shows that sets N_H can be used to describe the topouogy of $\boldsymbol{D}$.

We are now in a position to prove our first main theorem (from which Theorem I follows immediately):

THEOREM 2.2. *For $S \in \boldsymbol{E}'$ the following properties are equivalent:*

(a) *$\mathfrak{F}(S)$ is slowing decreasing.*

(b) *$S * f \to f$ is a continuous linear map of $S * \mathfrak{D} \to \mathfrak{D}$.*

(c) *$S * \mathfrak{D}' = \mathfrak{D}'$.*

(d) *There exists an invertible $S' \in \mathcal{E}'$ such that $S * T = S'$ has a solution $T \in \mathfrak{D}'$.*

(e) *$S * U \to U$ is a semi-continuous linear map of $S * \mathcal{E}' \to \mathcal{E}'$.* (That is, the map sends bounded sets into bounded sets.)

Proof. We shall prove Theorem 2.2 by the usual chain of implications: (a) implies (b), (b) implies (c), etc. We shall prove the simpler parts first.

(b) *implies* (c): This is an immediate consequence of the Hahn-Banach theorem.

(c) *implies* (d): This is a triviality.

(d) *implies* (e): Let $S * B$ be bounded. Then it follows that $S * B * T$ is bounded in $\mathfrak{D}'$, that is, $S' * B$ is bounded in $\mathfrak{D}'$ by the associativity and commutativity of convolution (see [24]). But all the $S * U$ for $U \in B$ have their carriers in a fixed compact set; hence, all $U \in B$ have their carrier in a fixed compact set by the theorem on addition of carriers (see [24]). Thus, $S' * B$ is a set which is bounded in $\mathfrak{D}'$ and all the distributions in $S' * B$ have their carriers in a fixed compact set. Thus (see [24]) $S' * B$ is bounded in $\mathcal{E}'$.

Now we can find a $T' \in \mathfrak{D}'$ such that $S' * T' = \delta$. Repeating the above argument we find that $\delta * B = B$ is bounded in $\mathcal{E}'$. This completes the proof that (d) implies (e).

We now have to prove the two difficult parts of our theorem:

Proof that (a) *implies* (b). We shall use the Lemmas 2.1 and 2.2 and Theorem 2.1. We assume first that $J=\mathfrak{J}(S)$ is bounded on R; we shall later dispense with this assumption. Let H be any function as described in Theorem 2.1. For each $z_0 \in C$ we choose an α satisfying the conclusions of Lemma 2.2; here we can take $p(z_0)=|\,\mathfrak{I}(z_0)|+A\log(1+|z_0|)$, and and $q(x)=(A+|x|)^{-A}$. In the notation of Lemma 2.1 we can choose therefore by Lemma 2.2,

$$(2.12)\qquad \begin{aligned} b(z_0) &= M(A+|x|)^{-dA}\exp(-dB(|\,\mathfrak{I}(z_0)|+A\log(1+|z_0|))) \\ &= M(A+|x|)^{-dA}(1+|z\ |)^{-daB}\exp(-dB\,|\,\mathfrak{I}(z_0)|). \end{aligned}$$

Since $|z_0-x|\leqq|\,\mathfrak{I}(z_0)|+A\log(1+|z_0|)$, we have

$$\begin{aligned} A+|x| &\leqq A+|z_0|+|z_0-x| \\ &\leqq A+2|z_0|+A\log(1+|z_0|) \\ &\leqq A'+3|z_0| \end{aligned}$$

for some A'. Thus, we can assume by changing M if necessary, that

$$(2.13)\qquad b(z_0)=M(M+|z_0|)^{-M}\exp(-M\,|\,\mathfrak{I}z_0|).$$

Thus, according to Lemma 2.1, we have to find an H' such that

$$(2.14)\qquad \max_{z\in\alpha} H'(z)\leqq M(M+|z_0|)^{-M}\exp(-M\,|\,\mathfrak{I}z_0|)H(z_0).$$

It is now clear that if H' exists then, by Lemma 2. 2, (2.9), a good choice would be

$$(2.15)\qquad H'(z)=\min_{|z-z_0|\leqq M|\,\mathfrak{I}z_0|+M\log(1+|z_0|)} M(M+|z_0|)^{-M}\exp(-M\,|\,\mathfrak{I}z_0|)H(z_0).$$

We have only to show that this choice of H' satisfies the hypotheses of Theorem 2.1. Let then $h(z)$ be a continuous function for which we can find an $a>0$ so that for all $m>0$, (2.10) is satisfied. To show that $h(z)/H'(z)$ is bounded, we have to show that if we define

$$(2.16)\qquad h'(z_0)=M^{-1}(M+|z_0|)^{M}\exp(M\,|\,\mathfrak{I}z_0|)\max_{|z-z_0|\leqq M|\,\mathfrak{I}z_0|+M\log(1+|z_0|)} h(z),$$

then $h'(z_0)$ again satisfies (2.10). Thus, we have only to show that

$$(2.17)\qquad h''(z_0)=\max_{|z-z_0|\leqq M|\,\mathfrak{I}z_0|+M\log(1+|z_0|)} h(z)$$

satisfies (2.10) whenever h does. To show that

$$h''(z_0)\exp(-M'\,|\,\mathfrak{I}z_0|)(1+|z_0|)^{m}$$

is bounded for a suitable M' is the same as showing that

$$(2.18)\qquad h(z)\max_{|z-z_0|\leqq M|\mathfrak{I}z_0|+M\log(1+|z_0|)}\exp(-M'|\mathfrak{I}z_0|)(1+|z_0|)^m$$

is bounded.

Consider $\min_{|z-z_0|\leqq M|\mathfrak{I}z_0|+M\log(1+|z_0|)}|\mathfrak{I}z_0|$. It is easily seen that if $|\mathfrak{I}z|\geqq 2M\log(1+|z|)$ then this minimum is $\geqq(4M)^{-1}|\mathfrak{I}z|$. Thus, if $|\mathfrak{I}z|\geqq 2M\log(1+|z|)$,

$$\max_{|z-z_0|\leqq M|\mathfrak{I}z_0|+M\log(1+|z_0|)}\exp(-M'|\mathfrak{I}z_0|)\leqq\exp(-M''|\mathfrak{I}z|)$$

for a suitable M''. Moreover, for all z, $\max\exp(-M'|\mathfrak{I}z_0|)\leqq 1$. Thus we can find an M''' so that for all z we have

$$(2.19)\qquad \max\exp(-M'|\mathfrak{I}z_0|)\leqq\exp(-M''|\mathfrak{I}z|)(1+|z|)^{M'''}.$$

Clearly we also have

$$(2.20)\qquad \max_{|z-z_0|\leqq M|\mathfrak{I}z_0|+M\log(1+|z_0|)}\exp(-|\mathfrak{I}z_0|)(1+|z_0|)^m\leqq m'(1+|z|)^{m'}$$

for a suitable m' depending only on m.

Putting (2.20), (2.19) together with (2.18) we have the desired result that $h''(z_0)$ satisfies (2.10) when h does. Thus H' defined by (2.15) satisfies the hypotheses of Theorem 2.1 and the desired implication (a) *implies* (b) is established when J is bounded on R.

In case J is not bounded on R, set $K(z)=\sin z_1\sin z_2\cdots\sin z_n/z_1z_2\cdots z_n$. Then for l large enough, $J'=K^lJ$ is bounded on R. Since the first partial derivatives of J on R are $O(1+|x|^p)$ for some p, we see easily that J' is again slowly decreasing. If $JF\to 0$ in $\boldsymbol{D}$, then so does $J'F$. Hence by the above, $F\to 0$ in $\boldsymbol{D}$. Thus, (a) *implies* (b) is proven in all cases.

(Instead of reducing the case of J unbounded to the case J bounded, we could have argued directly as we did in the bounded case.)

To complete the proof of Theorem 2.2 we have to show (e) *implies* (a): Assume then that J is not slowly decreasing; we have to find a set $B\subset\mathcal{E}'$ so that JB is bounded in $\mathcal{E}'$ but B is not bounded in $\mathcal{E}'$.

For JB to be bounded in $\mathcal{E}'$ all the functions $F\in B$ have to be bounded exponential type (say $\leqq\pi$). If we want B to be unbounded, then we want to make F large where J is small. Since J is small on large sets, such functions F can be constructed which are very large.

Since J is not slowly decreasing we can find a sequence of points $x^j\in R$ with the following properties:

α. $|x^j| > e^{3j}$.

β. On the set $\{y \in R, |y - x^j| \leqq j \log |x^j|\}$, we have $|J(y)| \leqq |y|^j$.

For each point x^j we want to find first an $F'_j \in E'$ so that $F'_j(x^j) \geqq (1/e)|x^j|^j$ but $|JF'_j(x)| \leqq 1$ for all $x \in R$. Moreover, we want to make sure that F'_j is of exponential type $\leqq \pi$.

We shall construct F'_j as $F'_j(z) = F_j(z_1)F_j(z_2) \cdots F_j(z_n)$, where $F_j \in {}_1\boldsymbol{D}$ (space $\boldsymbol{D}$ in one variable). Let us assume first that $n = 1$; the general case will be similar. Then we want $F_j(x^j) \geqq (1/e)|x^j|^j$ and then we want F_j to decrease very rapidly. By the minimum modulus theorem, F_j cannot decrease for a long distance faster than $\exp(-a(\text{distance to } x^j))$ for some $a > 0$. On the other hand, F_j cannot decrease exponentially for too long a distance, for this would contradict known inequalities on $\int_{-\infty}^{\infty} (\log |F_j(x)|/(1 + |x|^2))\,dx$. Thus, we want to construct F_j so that

a. $F_j(x^j) \geqq (1/e)|x^j|^j$.
b. $F_j(x)$ decreases exponentially as long as possible until it reaches a value $\leqq 1$.
c. $F_j(x)$ stays $\leqq 1$ after that point.
d. F_j is of exponential type $\leqq \pi$.

Let us define H_j by

$$H_j(z) = \prod_{k=1}^{\infty} (1 - z^2/j^2k^2)^j = ((j/\pi z)\sin(\pi z/j))^j.$$

Then the following properties are readily verified:

1. H_j is an entire function of exponential type π.
2. $H_j(0) = 1$.
3. $|H_j(x)| \leqq 1$ for x real.
4. $|H_j(x)| \leqq e^{-j}$ for $x \in R$, $|x| \geqq j$.

Then we set

$$F_j(z) = e^k H_k(z - x^j), \tag{2.21}$$

where k is the greatest integer in $j \log |x^j|$. Then $F_j(z)$ has the following properties:

1'. F_j is an entire function of exponential type π.
2'. $|F_j(x^j)| \geqq (1/e)|x^j|^j$.

is bounded for a suitable M' is the same as showing that

$$(2.18)\qquad h(z) \max_{|z-z_0|\leqq M|\mathfrak{I}z_0|+M\log(1+|z_0|)} \exp(-M'|\mathfrak{I}z_0|)(1+|z_0|)^m$$

is bounded.

Consider $\min\limits_{|z-z_0|\leqq M|\mathfrak{I}z_0|+M\log(1+|z_0|)} |\mathfrak{I}z_0|$. It is easily seen that if $|\mathfrak{I}z| \geqq 2M\log(1+|z|)$ then this minimum is $\geqq (4M)^{-1}|\mathfrak{I}z|$. Thus, if $|\mathfrak{I}z| \geqq 2M\log(1+|z|)$,

$$\max_{|z-z_0|\leqq M|\mathfrak{I}z_0|+M\log(1+|z_0|)} \exp(-M'|\mathfrak{I}z_0|) \leqq \exp(-M''|\mathfrak{I}z|)$$

for a suitable M''. Moreover, for all z, $\max\exp(-M'|\mathfrak{I}z_0|) \leqq 1$. Thus we can find an M''' so that for all z we have

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Clearly we also have

$$(2.20)\qquad \max_{|z-z_0|\leqq M|\mathfrak{I}z_0|+M\log(1+|z_0|)} \exp(-|\mathfrak{I}z_0|)(1+|z_0|)^m \leqq m'(1+|z|)^{m'}$$

for a suitable m' depending only on m.

Putting (2.20), (2.19) together with (2.18) we have the desired result that $h''(z_0)$ satisfies (2.10) when h does. Thus H' defined by (2.15) satisfies the hypotheses of Theorem 2.1 and the desired implication (a) *implies* (b) is established when J is bounded on R.

In case J is not bounded on R, set $K(z) = \sin z_1 \sin z_2 \cdots \sin z_n / z_1 z_2 \cdots z_n$. Then for l large enough, $J' = K^l J$ is bounded on R. Since the first partial derivatives of J on R are $O(1+|x|^p)$ for some p, we see easily that J' is again slowly decreasing. If $JF \to 0$ in $\boldsymbol{D}$, then so does $J'F$. Hence by the above, $F \to 0$ in $\boldsymbol{D}$. Thus, (a) *implies* (b) is proven in all cases.

(Instead of reducing the case of J unbounded to the case J bounded, we could have argued directly as we did in the bounded case.)

To complete the proof of Theorem 2.2 we have to show (e) *implies* (a): Assume then that J is not slowly decreasing; we have to find a set $B \subset \mathcal{E}'$ so that JB is bounded in $\mathcal{E}'$ but B is not bounded in $\mathcal{E}'$.

For JB to be bounded in $\mathcal{E}'$ all the functions $F \in B$ have to be bounded exponential type (say $\leqq \pi$). If we want B to be unbounded, then we want to make F large where J is small. Since J is small on large sets, such functions F can be constructed which are very large.

Since J is not slowly decreasing we can find a sequence of points $x^j \in R$ with the following properties:

α. $|x^j| > e^{3j}$.

β. On the set $\{y \in R, |y - x^j| \leqq j \log |x^j|\}$, we have $|J(y)| \leqq |y|^j$.

For each point x^j we want to find first an $F'_j \in E'$ so that $F'_j(x^j) \geqq (1/e)|x^j|^j$ but $|JF'_j(x)| \leqq 1$ for all $x \in R$. Moreover, we want to make sure that F'_j is of exponential type $\leqq \pi$.

We shall construct F'_j as $F'_j(z) = F_j(z_1)F_j(z_2) \cdots F_j(z_n)$, where $F_j \in {}_1\boldsymbol{D}$ (space $\boldsymbol{D}$ in one variable). Let us assume first that $n = 1$; the general case will be similar. Then we want $F_j(x^j) \geqq (1/e)|x^j|^j$ and then we want F_j to decrease very rapidly. By the minimum modulus theorem, F_j cannot decrease for a long distance faster than $\exp(-a(\text{distance to } x^j))$ for some $a > 0$. On the other hand, F_j cannot decrease exponentially for too long a distance, for this would contradict known inequalities on $\int_{-\infty}^{\infty}(\log|F_j(x)|/(1+|x|^2))dx$. Thus, we want to construct F_j so that

a. $F_j(x^j) \geqq (1/e)|x^j|^j$.
b. $F_j(x)$ decreases exponentially as long as possible until it reaches a value $\leqq 1$.
c. $F_j(x)$ stays $\leqq 1$ after that point.
d. F_j is of exponential type $\leqq \pi$.

Let us define H_j by

$$H_j(z) = \prod_{k=1}^{\infty}(1 - z^2/j^2k^2)^j = ((j/\pi z)\sin(\pi z/j))^j.$$

Then the following properties are readily verified:

1. H_j is an entire function of exponential type π.
2. $H_j(0) = 1$.
3. $|H_j(x)| \leqq 1$ for x real.
4. $|H_j(x)| \leqq e^{-j}$ for $x \in R$, $|x| \geqq j$.

Then we set

$$F_j(z) = e^k H_k(z - x^j), \tag{2.21}$$

where k is the greatest integer in $j \log |x^j|$. Then $F_j(z)$ has the following properties:

1'. F_j is an entire function of exponential type π.
2'. $|F_j(x^j)| \geqq (1/e)|x^j|^j$.

10

3′. $|F_j(x)| \leqq |x^j|^j$ for x real.

4′. $|F_j(x)| \leqq 1$ for $x \in R$, $|x - x^j| \geqq j \log |x^j|$.

Call $B = \{F_j\}$. Then the set B is not bounded in $\mathcal{E}'$ because of condition 2′. Consider $JB = \{JF_j\}$; I claim this set is bounded in $\mathcal{E}'$. Condition 1′ shows that all JF_j are of bounded exponential type. The fact that JB is bounded will be a consequence of the fact that

$$(2.22) \qquad |J(x)F_j(x)| \leqq 1 + |x| + |J(x)| \text{ for } x \in R,$$

as follows from our characterization of the bounded sets of $\boldsymbol{E}'$ (see [11]). To prove (2.22) we consider first those x for which $|x - x^j| \geqq j \log |x^j|$. For such x, inequality (2.22) is an immediate consequence of 4′. On the other hand, if $|x - x^j| \leqq j \log |x^j|$, then 3′ shows that $|F_j(x)| \leqq |x|^{-j}$. By our assumption α, $|x^j| \geqq e^{3j}$ so that if $|x - x^j| \leqq j \log |x^j|$, then $|x| \geqq \frac{1}{2} |x^j|$. Thus, for such x, we have $J(x) \leqq 2^j |x^j|^j$ so that

$$(2.23) \qquad |J(x)F(x)| \leqq 2^j \text{ for } |x - x^j| \leqq j \log |x^j|.$$

But, for such x, $2^j < |x|$ so (2.23) implies (2.22) which completes the proof that (e) *implies* (a) in case $n = 1$.

In case $n > 1$, we proceed exactly as above except that we replace the functions H_j used above by $H'_j(z) = H_j(z_1)H_j(z_2) \cdots H_j(z_n)$. We then define functions F'_j in terms of H_j exactly the way F_j was defined in terms of H_j above. The proof is then concluded exactly as in the case $n = 1$. This completes the proof of Theorem 2.2 and hence of Theorem I.

By a slight modification of the argument used in the proof that (e) *implies* (a), we can prove

PROPOSITION 2.3. *The conditions of Theorem 2.2 are equivalent to*

(f) *Let $U \subset \mathfrak{D}$ and $S * U$ be bounded in $\mathfrak{D}$; then U is bounded in $\mathcal{E}'$.*

PROPOSITION 2.4. *The conditions of Theorem 2.2 imply*

(g) *$S * \mathcal{E}'$ is bornologic.*

(h) *$S * \mathfrak{D}$ is bornologic.*

Proof. We prove the part of the theorem for $\mathfrak{D}$, as the part of the theorem concerning $\mathcal{E}'$ is handled similarly, except that we have to use the fact that condition (d) of Theorem 2.2 implies that $S * U \to U$ is a continuous map of $S * \mathcal{E}' \to \mathcal{E}'$ (see Proposition 2.7 below).

Assume that S satisfies the conditions of Theorem 2.2; let L be a linear function on $S * \mathfrak{D}$ which is bounded on the bounded sets. Define T on $\mathfrak{D}$ by

$$T \cdot h = L \cdot S * h \text{ for } h \in \mathfrak{D}.$$

Then by Theorem 2.2, T is bounded on the bounded sets of $\mathfrak{D}$; since $\mathfrak{D}$ is bornologic, T is continuous on $\mathfrak{D}$, that is, T is a distribution.

Now, let $S * h \to 0$ in the topology of $\mathfrak{D}$; by Theorem 2.2 it follows that $h \to 0$ in the topology of $\mathfrak{D}$. Thus, $T \cdot h \to 0$ so L is continuous on $S * \mathfrak{D}$. This proves that $S * \mathfrak{D}$ is bornologic.

Remark. I do not know if (g) and (h) are true for any S (not necessarily invertible) or whether they imply S is invertible.

THEOREM 2.5. *The conditions of Theorem* 2.2 *are equivalent to*

(i) $S * \mathfrak{D}' \supset \mathfrak{D}$.

Proof. It is clear that (c) implies (i). Assume then that S is not invertible; I shall construct an $f \in \mathfrak{D}$ which is not in $S * \mathfrak{D}'$.

Let us note the following: Let B be a set in $\mathfrak{D}$ for which $S * B$ is bounded in $\mathfrak{D}$. Then if $S * W = f$ for $W \in \mathfrak{D}'$, it must be the case that

$$W \cdot S * h = S * W \cdot h = f \cdot h$$

is uniformly bounded for $h \in B$. Thus, to prove Theorem 2.5 we must produce a set $B \subset D$ with $S * B$ bounded in $\mathfrak{D}$ but $\{f \cdot h\}_{h \in B}$ not bounded. Proposition 2.3 shows that there is hope for this because we can choose B not bounded in $\mathcal{E}'$ with $S * B$ bounded in $\mathfrak{D}$. However, B being not bounded in $\mathcal{E}'$, it follows (see [4]) that B is not weakly bounded in $\mathcal{E}'$. Hence, there exists an $f' \in \mathcal{E}$ so that $\{f' \cdot h\}_{h \in B}$ is not bounded. But the functions $h \in B$ have their carriers in a fixed compact set $K \subset \mathfrak{D}$. Hence, if $f'' \in \mathfrak{D}$ is 1 on K, for any $h \in B$ we have $f' \cdot h = f'f'' \cdot h$. But $f = f'f'' \in \mathfrak{D}$; hence we have $\{f \cdot h\} = \{f' \cdot h\}$ is unbounded which concludes the proof of the theorem.

Remark. Using the notations of the last part of the proof of Theorem 2.2 above ((e) *implies* (a)), we could also write f explicitly (or rather its Fourier transform F) in the form

$$F(z) = \sum c_j H(z - x^j) F'_j(z),$$

where $H \in D$, $H(0) = 1$, $0 \leqq H(x) \leqq 1$ for $x \in R$, and where the c_j are suitably chosen constants.

Theorem 2.5 settles a problem of the author (see [7]), namely, that for $f \in \mathfrak{D}$, $f * \mathfrak{D}' \neq E$. However, I do not know if $\mathfrak{D} * \mathcal{E} = \mathcal{E}$ or even if $\mathfrak{D} * \mathfrak{D}' = \mathcal{E}$, although even $\mathfrak{D} * \mathcal{E} = \mathcal{E}$ is undoubtedly true.

Another interesting question in this connection was raised by Professor

Chevalley in his lectures on the theory of distributions: Is $\mathfrak{D} * \mathfrak{D} = \mathfrak{D}$? This problem seems very difficult. (See appendix at end of paper.)

THEOREM 2.6. *The conditions of Theorem 2.2 are equivalent to*

(j) *For any entire function G, if $JG \in \boldsymbol{D}$ (or $JG \in \boldsymbol{E}'$), then $G \in \boldsymbol{D}$ (resp. $G \in \boldsymbol{E}'$).*

In fact, for (j) *to hold it is sufficient that $JG \in \boldsymbol{D}$ should imply $G \in \boldsymbol{E}'$.*

Proof. If J is slowly decreasing, then by applying the minimum modulus theorem in the manner used in proving Theorem 2.2, (a) *implies* (b), we can show that (j) holds.

Conversely, suppose that S is not invertible; I shall produce an entire function G (which is necessarily of exponential type) such that $JG \in \mathfrak{D}$ but $G \notin \boldsymbol{E}'$. For this purpose we revert to the notation of the proof of Theorem 2.2, (e) implies (a). Let $H \in \boldsymbol{D}$ be so chosen that $H(0) = 1$, $0 \leqq H(x) \leqq 1$ for $x \in R$. I want to write first

$$G(z) = \sum c_j H(z - x^j) F'_j(z) \tag{2.24}$$

for suitable constants c_j. Now, it is clear from the construction that I can assume that the x^j are chosen so large that the intervals

$$\{x \mid x \in R, \mid x - x^j \mid \leqq j \log \mid x^j \mid\} \tag{2.25}$$

do not overlap. Then we choose $c^j = j^{-2} \mid x_j \mid^{-j/2}$.

It is clear from (2.24) and (2.25) that the series for G' converges uniformly on the compact sets of R; moreover, for any j,

$$\begin{aligned} G(x^j) &= \sum c_k F'_k(x^j) \\ &\geqq F'_j(x^j) - \sum_{k \neq j} c_k F'_k(x^j) \\ &\geqq (1/ej^2) \mid x^j \mid^{j-j/2} - \sum p^{-2} \\ &\geqq (1/2e) \mid x^j \mid^{j/2} \end{aligned} \tag{2.26}$$

for j sufficiently large because of condition α on the choice of the x^j. Thus, if G is an entire function, it is certainly not in $\boldsymbol{E}'$.

Next, call $K(z) = (\sin z_1/z_1)(\sin z_2/z_2) \cdot \cdot \cdot (\sin z_n/z_n)$. For l sufficiently large, $J'(z) = K^l(z) J(z)$ is bounded on R. Since K^l is clearly slowly decreasing, it is sufficient by means of the proof of the first part of this theorem (i.e., that S invertible implies (j)) to prove our result for J' in place of J; that is, we may assume J is bounded on R.

By use of the method of proof of inequality (2.22) above, it follows that

there exists an $a > 0$ so that for any j,

$$\sup_{x \in R} c_j \, | F'_j(x) J(x) | \leqq a/2^j. \tag{2.27}$$

Since all F'_j are entire functions of exponential type $\leqq \pi$, this shows (see [11]) that the series

$$\sum c_j J(F'_j(z) H(z - x^j)) \tag{2.28}$$

converges in the topology of $\boldsymbol{E}'$. It is easy to see, in fact, using the characterization of the topology of $\boldsymbol{D}_l$ that this series converges in $\boldsymbol{D}_l$ for l large enough if the x^j are sufficiently large.

Thus, in particular, the series (2.28) converges in the topology of the space H_l of entire functions on C of exponential type $\leqq i$ for l large enough (see [8]). Using the minimum modulus theorem it follows easily that the series $\sum c_j F_j(z) H(z - x^j)$ also converges in H_l for l large enough. Thus, G is an entire function of exponential type.

In resumé, G is an entire function which is not in $\boldsymbol{E}'$, but $JG \in \boldsymbol{D}$. This completes the proof of Theorem 2.6.

PROPOSITION 2.7. *The conditions of Theorem 2.2 are equivalent to*

(k) *$S * W \to W$ is a continuous linear map of $S * \mathcal{E}' \to \mathcal{E}'$.*

(l) *$S * \mathcal{E} = \mathcal{E}$.*

Proof. The equivalence of (d), (k), and (l) is a fairly simple consequence of the Hahn-Banach and closed graph theorems and was established by Malgrange in [21].

In all the above we were concerned with invertible operators for the spaces $\mathcal{E}$ and $\mathcal{D}'$; we wish here to give the analogous description for the space $\mathcal{D}'_F$ (see [5], [21]). Let $J \in \boldsymbol{E}'$; J is called *very slowly decreasing* if there exists an $A > 0$ so that for any $x \in R$ we can find a $y \in R$ with $|y - x| \leqq A$ and $|J(y)| \geqq (A + |x|)^{-A}$. Then we have

THEOREM 2.2*. *The following conditions are equivalent for $S \in \mathcal{E}'$:*

(a*) *J is very slowing decreasing.*

(b*) *$S * f \to f$ is a continuous linear map of $S * \mathcal{D}_F \to \mathcal{D}_F$.*

(c*) *$S * \mathcal{D}'_F = \mathcal{D}'_F$.*

(d*) *There exists an $S' \in \mathcal{E}'$ which is invertible for $\mathcal{D}'_F$ such that $S * T = S'$ has a solution $T \in \mathcal{D}'_F$.*

(e*) *For each $m > 0$ there is an $r > 0$ so that if B is a subset of $\mathcal{D}$ for which $S * B$ is bounded in $\mathcal{D}^r$, then B is bounded in $\mathcal{D}^m$.*

The proof of Theorem 2.2* is very similar to the proof of Theorem 2.2 and so will be omitted. The equivalence of (c*) and (d*) was proven by Malgrange in [21].

Remark. I have not been able to construct an $S \in \mathcal{E}'$ which is invertible for $\mathfrak{D}'$ but not for $\mathfrak{D}'_F$.

All the above has concerned itself with the solution of the question of when $S * \mathfrak{D}' = \mathfrak{D}'$ (or $S * E = E$, etc.). We could ask the question as to what is $S * \mathfrak{D}'$ even in case S is not invertible. As mentioned in the introduction, this involves describing explicitly the topology τ on $\mathfrak{D}$ which is defined by: N is a neighborhood of zero in τ if $S * N$ is a neighborhood of zero in $S * \mathfrak{D}$. That is, τ is the strongest topology so that the map $S * f \to f$ of $\mathfrak{D} \to \tau$ is continuous. Of course, in case S is invertible, then τ coincides with $\mathfrak{D}$.

Actually, we are not able to give a "good" description of τ; this seems to be because we have not been able to prove that the spaces $S * \mathfrak{D}$ are bornologic even if S is not invertible. However, we shall give instead the description of the restriction of τ to each $S * \mathfrak{D}_l$. We denote this restriction again by τ.

Of course, we want another expression for the topology τ, one which does not depend so much on S, and one which is useful. We shall give instead the topology σ of the Fourier transform of τ in a form which will be suitable for our purposes. For this we define functions $M_l(z)$ on C which are certain majorants of $J(z)$. For any $z \in C$ and any $l > 5$ (*exponential type* J) we set

$$(2.29) \qquad M_l(z) = \max_{z' \in C,\, |\mathfrak{I}(z')| \leqq \mathfrak{I}(z)} \exp(-l\,|\,z' - z\,|)\,|\,J(z')\,|.$$

We shall describe the topology σ using M_l instead of J. This is a great advantage because M_l behaves much more regularly than J, and the zeros of J do not enter into the M_l.

THEOREM 2.8. *For each $l' > 0$ the topology σ on $\mathfrak{D}_{l'}$ can be described as follows: Let l be a fixed number $> l' + d$, where d is some number depending only on the exponential type of J. For each integer $m > 0$ call N the set of $F \in \mathfrak{D}_l$ such that*

$$(2.30) \qquad |\,z^m{}_k F(z)\,| \leqq \exp(l\,|\,\mathfrak{I}(z)\,|)$$

for $k = 1, 2, \cdots, n$. Then these sets form a fundamental system of neighborhoods of zero for σ.

Proof. As in the proof of Theorem 2.6 we may assume $|\,J(x)\,| \leqq 1$ for

$x \in R$. Since $M_l(z) \geqq J(z)$ for all z, we have only to prove that the sets N are neighborhoods of zero for τ.

Let $z \in C$ be fixed; let z' be a point with $\Im z' = \Im z$, where $M_l(z) = \exp(-l \mid z' - z \mid) \mid J(z') \mid$. We shall assume first that $n = 1$. Suppose that we had

$$(2.31) \qquad \mid z''^m F(z'') J(z'') \mid \leqq \exp(l' \mid I(z'') \mid$$

for all $z'' \in C$. Now, by the minimum modulus theorem, we can draw about the point z' a circle γ of radius between $\mid z' - z \mid$ and $2 \mid z' - z \mid$ so that for all $z'' \in \gamma$ we have for certain constants c and d which depend only on the exponential type of J,

$$(2.32) \qquad \mid J(z'') \mid \geqq c \exp(-d \mid z' - z \mid) \mid J(z') \mid \exp(-d \mid \Im z \mid).$$

Combining (2.31) and (2.32) we have for all $z'' \in \gamma$,

$$(2.33) \qquad \mid J(z') F(z'') z''^m \mid \leqq (1/c) \exp(d \mid z' - z \mid + (l' + d) \mid \Im z \mid).$$

Since $F(z'')z''^m$ is an entire function of z'' we have, by the maximum modulus theorem,

$$(2.34) \qquad \mid J(z') F(z) z^m \mid \leqq (1/c) \exp(d \mid z' - z \mid + (l' + d) \mid \Im z \mid)$$

or, what is the same thing,

$$(2.35) \qquad \mid J(z') \exp(-d \mid z' - z \mid) F(z) z^m \mid \leqq (1/c) \exp[(l' + d) \mid \Im z \mid].$$

Now, if l is larger than d,

$$(2.36) \qquad M_l(z) = \exp(-l \mid z' - z \mid) \mid J(z') \mid \leqq \exp(-d \mid z' - z \mid) \mid J(z') \mid.$$

Thus (2.35) implies

$$(2.37) \qquad \mid M_l(z) F(z) z^m \mid \leqq (1/c) \exp[(l' + d) \mid \Im z \mid]$$

which gives our result in case $n = 1$.

The case $n > 1$ is handled by the same method except that we apply the minimum and maximum modulus theorem in each variable separately. We shall omit the details.

All that has been done previously in regard to the invertible operators is in connection with the question of when $S * \mathfrak{D}' = \mathfrak{D}'$. On the other hand, we might ask when does $S * \mathfrak{D}' \supset T * \mathfrak{D}'$, where T is a distribution in $\mathcal{E}'$? Call J the Fourier transform of S and K the Fourier transform of T. We might expect that $S * \mathfrak{D}' \supset T * \mathfrak{D}'$ should be equivalent to the fact that K/J does not tend to zero too fast at infinity. However, we are not able to establish

this fact; this problem seems to be essentially the same as the problem of describing the topology σ on $\boldsymbol{D}$ itself which, as we mentioned, we are not able to accomplish. However, we can prove part of the analogue for the spaces D'_l. For this purpose we make the following

Definition. We say that J/K is *slowly decreasing* if for each l sufficiently large there exists a j so that for all $z \in C$,

$$M_l(K;z)/M_l(J;z) \leqq j(1+|Rz|)^j \exp(d\,|\mathfrak{J}z|), \tag{2.38}$$

where $d = 100n$ $(exp.\,type\,J + exp.\,type\,K + 1)$. (Here we have written $M_l(J;z)$, $M_l(K;z)$ to avoid confusion.)

It is easily seen that $J/1$ is slowly decreasing in the above sense if and only if J is slowly decreasing in our previous sense.

We can now formulate a partial extension of Theorem 2.2:

THEOREM 2.9. *For $S, T \in \mathcal{E}'$ each property implies the succeeding one:*

(a′) *J/K is slowly decreasing.*

(b′) *$S * f \to T * f$ is a continuous linear map of $S * \mathfrak{D}_k \to T * D_k$ for each k.*

(c′) *For each k there exists a k' so that $S * \mathfrak{D}'_{k'} \supset T * \mathfrak{D}'_k$.*

(d′) *For each k there exists a $W \in \mathfrak{D}'_k$ such that $S * W = T$.*

(e′) *$S * U \to T * U$ is a semi-continuous linear map of $S * \mathcal{E}' \to T * \mathcal{E}'$.*

Proof. As in the proof of Theorem 2.2, the implications (b′) *implies* (c′), (c′) *implies* (d′), and (d′) *implies* (e′) are easy. Moreover, (a′) *implies* (b′) is an easy consequence of Theorem 2.8.

We are not able to prove (e′) implies (a′). However, we can prove

THEOREM 2.10. *If condition* (e′) *holds, then for any $\epsilon > 0$ we can find a j (possibly depending on ϵ) so that*

$$[M_l(K,z)]^{1+\epsilon}/M_l(J,z) \leqq j(1+|Rz|)^j \exp(d'\,|\mathfrak{J}z|). \tag{2.39}$$

(Here $d' = d + \pi/\epsilon$.)

Proof. Assume (e′) holds but (2.39) does not hold. Then we can find an $\epsilon > 0$ and a sequence of points z^j with $z^j \to \infty$ fast enough so that

$$[M_l(K,z^j)]^{1+\epsilon}/M_l(J,z^j) > j(1+|Rz^j|)^j \exp(d'\,|\mathfrak{J}z|). \tag{2.40}$$

As in the proof of Theorem 2.6 we may assume $|K(x)| \leqq 1$ and $|J(x)| \leqq 1$ for $x \in R$.

We shall show first that we may assume that z^j are so chosen that $M_l(K,z^j)=|K(z^j)|$. For this purpose, assume (2.40) holds and let w be chosen so that $M_l(K,z^j)=\exp(-l\,|z^j-w|)\,|K(w)|$. Then clearly

$$M_l(K,w)\geqq K(w)=\exp(l\,|z^j-w|)M_l(K,z^j). \tag{2.41}$$

I claim that we must have $M_l(J,w)\leqq\exp(l\,|z^j-w|)M_l(J,z^j)$. For, assume this is not the case; let v be chosen so that $M_l(J,z^j)=\exp(-l\,|z^j-v|)J(v)$. Then if $M_l(J,w)>\exp(l\,|z^j-w|)M_l(J,z^j)$, we would have by the triangle inequality

$$\begin{aligned}M_l(J,w)&>\exp(l\,|z^j-w|)\exp[-l(|z^j-v|)]J(v)\\&\geqq\exp(-l\,|w-v|)J(v)\end{aligned}$$

which contradicts the definition of $M_l(J,w)$. Moreover, this argument shows that we have equality in (2.41).

Thus we have shown that

$$[M_l(K,z^j)]^{1+\epsilon}/M_l(J,z^j)\leqq([M_l(K,w)]^{1+\epsilon}/M_l(J,w))\exp(-\epsilon l\,|z^j-w|)$$

which implies that (2.40) holds with w in place of z^j. Hence, we may assume that z^j satisfies

$$M_l(K,z^j)=|K(z^j)|. \tag{2.42}$$

Next, I want to obtain an estimate for a cube containing z^j so that for all points z with $\mathfrak{J}z=\mathfrak{J}z^j$ in this cube we have

$$|J(z)|\leqq|K(z^j)|\,j^{-1/2}\exp(-d'\,|\mathfrak{J}z^j|/2)(1+|\mathfrak{R}z^j|)^{-j/2}.$$

Let w be a point for which $\mathfrak{J}w=\mathfrak{J}z^j$ and

$$|J(w)|>|K(z^j)|\,j^{-1/2}\exp(-d'\,|/2)\cdot(1+|\mathfrak{R}z^j|)^{-j/2}.$$

Then we estimate $M(J;z^j)$ as follows:

$$\begin{aligned}M_l(J;z^j)&\leqq\exp(-l\,|z^j-w|)\,|J(w)|\\&>\exp(-l\,|z^j-w|)K(z^j\,|\,j^{-1/2}\exp(-d'\,|\mathfrak{J}z^j|/2)(1+\mathfrak{R}z^j|)^{-j/2}.\end{aligned}$$

On the other hand, we know by (2.40) that

$$M_l(J;z^j)<|K(z^j)|^{1+\epsilon}/j(1+|\mathfrak{R}z^j|)^j\exp(d'\,|\mathfrak{J}z^j|).$$

Thus we must have

$$|K(z^j)|\exp(-l\,|z^j-w|)<|K(z^j)|^{1+\epsilon}/j^{1/2}(1+|\mathfrak{R}z^j|)^{j/2}\exp(d'\,|\mathfrak{J}z^j|/2).$$

Hence,

$$-l\,|\,z^j - w\,| < \epsilon \log |\,K(z^j)\,| - \tfrac{1}{2}\log j - (j/2)\log(1 + |\,\Re z^j\,|) - d'\,|\,\Im z^j\,|/2.$$

or

$$(2.43)\qquad \begin{aligned} |\,z^j - w\,| > &- (\epsilon/l)\log|\,K(z^j)\,| + \tfrac{1}{2}\log j \\ &+ (j/2)\log(1 + |\,\Re z^j\,|) + d'\,|\,\Im z^j\,|/2. \end{aligned}$$

As in the proof of Theorem 2.2, (e) *implies* (a), the proof for general n is similar to the proof for $n = 1$ which we shall henceforth assume. We shall also use the notation of the proof of Theorem 2.2, (e) *implies* (a). We may clearly assume for simplicity that $\Im z^j \geqq 0$ for all j and that $0 < \epsilon < 1$. Then we set

$$(2.44)\qquad F''_j(z) = e^m H_m((z - z^j)/\epsilon)\exp[-i(\pi + 1)(z - z^j)/\epsilon],$$

where

$$(2.45)\quad m = [-\log|\,K(z^j)\,| + d'\,|\,\Im z^j\,|/2 + \tfrac{1}{2}\log j + (j/2)\log(1 + |\,\Re z^j\,|)],$$

the bracket denoting as usal the integral part. Note that since $|\,K(x)\,| \leqq 1$ for $x \in R$, $-\log|\,K(z^j)\,| + d'\,|\,\Im z^j\,|/2 > 0$. Then we have the following properties:

1″. F''_j is an entire function of exponential type $2(\pi + 1)/\epsilon$.

2″. $|\,K(z^j)F''_j(z^j)\,| \geqq (1/e)j^{1/2}(1 + |\,\Re z^j\,|^{j/2})\exp(d'\,|\,\Im z^j\,|/2)$.

3″. $|\,F''_j(z)\,| \leqq |\,K(z^j)\,|^{-1}\exp(d'\Im z^j/2)(j/2)(1 + |\,\Re z^j\,|)^{j/2}$ for $Iz = Iz^j$.

4″. $|\,F''_j(z)\,| \leqq 1$ for $\Im z = \Im z^j$,

$$(2.46)\qquad \begin{aligned} |\,z - z^j\,| \geqq \epsilon(&-\log|\,K(z^j)\,| + d\,|\,\Im z^j\,|/2 \\ &+ \tfrac{1}{2}\log j + (j/2)\log(1 + |\,\Re z^j\,|). \end{aligned}$$

Conditions 1″ and 2″ show that $\{KF''_j\}$ is not bounded in $\mathcal{E}'$. Condition 3″ together with the argument following (2.42) shows that $|\,J(z)F''_j(z)\,| \leqq 1$ for $\Im z = \Im z^j$ and

$$(2.47)\qquad \begin{aligned} |\,z - z^j\,| \leqq &- (\epsilon/l)\log|\,K(z^j)\,| \\ &+ \tfrac{1}{2}\log j + (j/2)\log(1 + |\,\Re z^j\,|) + d'\,|\,\Im z^j\,|/2. \end{aligned}$$

Condition 4″ shows that for z satisfying (2.46) we have

$$|\,J(z)F''_j(z)\,| \leqq \exp(d_0\,|\,\Im z^j\,|) \qquad (d_0 = \exp\operatorname{type} J).$$

Hence, by the Phragmén-Lindelöf theorem, we have for all $x \in R$ (since we may assume $\epsilon^{-1} > d_0 = \exp\operatorname{type} J$) $|\,J(x)F''_j(x)\,| \leqq 1$. This proves that $\{JF''_j\}$ is bounded in $\mathfrak{D}_k$ for k large enough, which concludes the proof of Theorem 2.10.

Remark 1. By a slight modification of the above process we could prove that if (2.39) does not hold, then $S * f \to T * f$ is not a semi-continuous map of $\mathcal{D}$ into $\mathcal{E}'$.

Remark 2. In case $K = 1$, of course, we can replace $[M(K;z)]^{1+\epsilon}$ by $[M(K;z)]$ because $M(K;z)$ is itself $> (j + |\mathfrak{R}z|^{-j}\exp(-d\,|\mathfrak{J}z|)$ for some j. Thus the conditions (a′), (b′), (c′), (d′), (e′) are equivalent (and this fact is, of course, much weaker than Theorem 2.2). But the arguments used above can show that in this case, if there exists a distribution $W \in \mathcal{D}'$ with $S * W = \delta$ on $\mathcal{D}_k$, then S is invertible.

Remark 3. We have taken only one possible choice for the majorants M_l; actually, many possibilities present themselves. For example, we could replace the right side of (2.29) by

$$M_l(J, \lambda; z) = \max_{z' \in C,\, |\mathfrak{J}(z')| \leqq |\mathfrak{J}(z)| + \lambda} \exp(-l\,|z' - z|)\,|J(z')|, \tag{2.48}$$

where $\lambda \geqq 0$ is suitably chosen. All the above theorems would then be proven with no essential modification. The above generalization has the advantage that $(n = 1)$ the ratio of majorants

$$M_l(J', \lambda; z)/M_l(J, \lambda + 1; z) \leqq e^l \tag{2.49}$$

for any J. This follows immediately from the above and Cauchy's formula for the derivative of an analytic fuction. The result (2.49) seems of great importance in understanding the deeper parts of the theory of mean periodic functions for it shows that (in a slightly broader sense than used above) J/J' is slowly decreasing. (Compare Section 3 below and [27], particularly the latter where the properties of the ratio J'/J are of great importance.).

3. Invertibility and mean periodic functions. We are now going to study the relationship of the above with L. Schwartz' theory of mean periodic functions (see [25], [26], [27], [8]). We assume that $n = 1$ in the following because Schwartz' mean periodic expansion holds only in this case except for some special cases in higher dimension (see [15], [16]). We shall first briefly recall the main aspects of this theory:

Let V be a closed linear subset of $\mathcal{E}$ which is closed under translation (and hence also convolution by elements of $\mathcal{E}'$). We want to expand a given function $f \in V$ in terms of the exponential polynomials which belong to V; in particular, we wish to show that every $f \in V$ is the limit of the exponential polynomials of V. Assume V is not all of $\mathcal{E}$; then there exists an $S \in \mathcal{E}'$ satisfying $S \cdot f = 0$ for all $f \in V$. Since V is closed under translation we also

have $S * f = 0$ for all $f \in V$. In particular, the Fourier transform J of S must vanish at each point $z \in C$ for which $\exp(iz\cdot) \in V$ and J must have a zero at z of order at least $j+1$ if $x^j \exp(iz \cdot x) \in V$.

Now, suppose we had some expansion

$$(3.1)\qquad f(x) \sim \sum_{j,k} c_{jk} x^j \exp(iz^k \cdot x),$$

where the sum is taken over all pairs j, k for which $x^j \exp(iz^k \cdot x) \in V$.

We denote by $\{1/J(z)\}_{z^k}$ the principal part of the expansion of $1/J(z)$ at z^k; we set $J_k(z) = J(z)\{1/J(z)\}_{z^k}$ and we call S_k the Fourier transform of J_k. Then a simple computation shows that if (3.1) holds and if we have some kind of convergence, then (see [27])

$$(3.2)\qquad (S_k * f)(x) = \sum_{j=0}^{j'-1} c_{jk} x^j \exp(iz^k \cdot x),$$

where j' is the order of the zero of S at z^k. Thus, formally,

$$(3.3)\qquad f = \sum f * S_k = f * \sum S_k$$

provided the sum $\sum S_k$ converges in an appropriate sense. Now, (3.3) would hold for all $f \in V$, or even for all f which satisfy $S * f = 0$ provided that we could demonstrate the existence of a $T \in \mathcal{E}'$ with

$$(3.4)\qquad \delta = \sum S_k + S * T.$$

The Fourier transform of this relationship is

$$(3.5)\qquad 1 = \sum J\{1/J\}_{z^k} + JK,$$

where K is the Fourier transform of T. Let us note the following:

$$(3.6)\qquad (d^p/dz^p)[J\{1/J\}_{z^k}](z^l) = \begin{cases} 0 \text{ if } k \neq l,\ 0 \leqq p \leqq j'_l + 1 \\ 1 \text{ if } k = l,\ p = 0 \\ 0 \text{ if } k = l,\ 1 \leqq p \leqq j'_k + 1 \end{cases}$$

because $1 = J\{1/J\} + J \cdot$ regular part. Thus it follows that J divides $1 - \sum J\{1/J\}_{z^k}$ in the ring of entire functions. Now, suppose we can show $\sum J\{1/J\}_{z^k}$ belongs to $\boldsymbol{E}'$; then if J is slowly decreasing, the existence of K will be verified by Theorem 2.6.

Thus, if J is slowly decreasing, the whole theory of mean periodic functions will have a very simple structure. In case J is not slowly decreasing then it does not seem that formulae like (3.1) or (3.3) can hold; rather we shall show that they hold only in a certain limit sense.

Our main job is to show:

Suppose J is slowly decreasing. Then it is possible to find groupings $G_1, G_2, \cdots$ of the points z^k so that the series

$$\sum_{r=1}^{\infty} \sum_{z^k \in G_r} J\{1/J\}_{z^k} \tag{3.7}$$

converges in the topology of $\mathbf{E}'$.

This statement is not quite true (or, at least, I cannot prove it), and we shall derive a slightly modified form (Theorem 3.1 below). Following the method of Schwartz (see [25]) we write for $z \neq z^k$

$$(\{1/J\}_{z^k})(z) = \int_{\Gamma_k} d\zeta/J(\zeta)(z-\zeta),$$

where Γ_k is a closed curve containing z^k in its interior but not containing z or any $z^{k'}$ for $k' \neq K$. Hence, if $z \neq z^k$ for any $z^k \in G_r$,

$$\sum_{z^k \in G_r} \{1/J\}_{k_z} = \int_{\Gamma_r} d\zeta/J(\zeta)(z-\zeta), \tag{3.8}$$

where now Γ_r is a closed curve containing all $z^k \in G_r$ but not containing z or any $z^{k'} \notin G_r$.

The fact that z does not lie in any Γ_r is of no consequence for the convergence of (3.7) because if z lies in Γ_r, then we would get a contribution of $1/J(z)$ (if $J(z) \neq 0$) to the integral in (3.8). Since we are going to multiply by $J(z)$ anyway, this does not affect the convergence. Thus, I want to find a sequence of groupings G_r so that I can prove the series

$$\sum_{r=1}^{r} \int_{\Gamma_r} d\zeta/J(\zeta)(z-\zeta) \tag{3.9}$$

converges in a suitable sense. It is clear that the contours Γ_r have to be chosen in such a way that J is large on Γ_r; the possibility of choosing such Γ_r depends on the fact that J is slowly decreasing.

I shall assume first that all the zeros of J are real; we shall explain later how the former restriction is removed. Now, since J is slowly decreasing, there exists a positive integer j so large that for each $x \in R$ there is a $y \in R$ with $|y-x| \leqq j\log(j+|x|)$ and $|J(y)| \geqq (j+|y|)^{-j}$. For each integer k, positive or negative or zero, let A_k be the interval

$$A_k = \{x \in R \mid |x-k| \leqq j\log(j+|k|+2)\} \tag{3.10}$$

By the above, in A_k there is a point y_k with $|J(y_k)| \geqq (j+|y_k|)^{-j}$.

Now we are in a position to apply the minimum modulus theorem to

construct Γ_k. About y_k we can draw a circle Γ'_k of center y_k, radius R_k such that

$$(3.11)\qquad 4j\log(j+|k|+2) \leqq R_k \leqq 8j\log(j+|k|+2),$$

so that for all points $z \in \Gamma'_k$ we have

$$(3.12)\qquad |J(z)| \geqq (l+|y_k|)^{-l}$$

for some $l>0$ which depends only on J (see Theorem 5 of [8], p. 317). We need a slight sharpening of this estimate: Not only does (3.12) hold for all points on Γ'_k but we can find a number $q>0$ depending only on J so that (3.12) holds for all points z with

$$(3.13)\qquad R_k - q \leqq |z-y_k| \leqq R_k + q.$$

The proof of this can be obtained by a slight modification of the proof of Theorem 5 of [8], p. 317.

We notice that if we replace J by $J'(z)=J(z)z^{l+d}$ (d sufficiently large) then inequality (3.12) can be improved to

$$(3.14)\qquad J'(z) \geqq c(1+|y_k|)^d$$

for all z satisfying (3.13).

Now, we are ready to define the curves Γ_r (which depend slightly on z). we set $\Gamma_0=\Gamma'_0$ unless $||z|-|R_0||<q$ in which case we replace Γ'_0 by a circle of radius R'_k between R_k-q and R_k+q so that $||z|-|R'_k||<q$. Suppose Γ_r have been defined for $r=0,\pm 1,\pm 2,\cdots,\pm r'$. Then de define $\Gamma_{r'+1}$ as follows: ($\Gamma_{r'-1}$ is defined similarly)

1. If $y_{r'+1}$ is contained in or on $\bigcup_{|r|\leqq r'}\Gamma_r$, then $\Gamma_{r'+1}$ is empty.

2. If $y_{r'+1}$ is not contained in or on $\bigcup_{|r|\leqq r'}\Gamma_r$ and if $||z-y_{r'+1}|-R_{r'+1}| \geqq q$, then $\Gamma'_{r'+1}$ intersects $\bigcup_{|r|\leqq r'}\Gamma_r$ and we choose that connected component of $\Gamma'_{r'+1}$ *minus* this intersection which meets the real axis at a point $>y_{r'+1}$. $\Gamma_{r'+1}$ is the union of this component with arcs of $\bigcup_{|r'|\leqq r}\Gamma_r$, these arcs being chosen in such a manner that $\Gamma_{r'+1}$ is closed, simple, and does not contain in its interior any points which lie in the interior of some Γ_r for $|r|\leqq r'+1$. It is easily seen that this curve is uniquely determined by our description.

3. If $y_{r'+1}$ is not contained in or on $\bigcup_{|r|\leqq r'}\Gamma_r$ and if $||z-y_{r'+1}|-R_{r'+1}| < q$, then we choose an $R'_{r'+1}$ so that $||z-y_{r'+1}|-R'_{r'+1}|\geqq q$ and $|R'_{r'+1}-R_{r'+1}|\leqq q$ and we proceed as in 2 above.

Now, the number of circles Γ'_r that $\Gamma'_{r'}$ can meet for $|r| < |r'|$ is certainly $< 2r'$. Since R_r satisfies (3.13), it follows easily that the length of $\Gamma_{r'}$, which cannot exceed the sum of the circumferences of circles of radii $R_r + q$, must be $\leqq \text{const}(1 + |r'|)^2$.

We can now prove

LEMMA 3.1. *The series* $\sum_{r=-\infty}^{\infty} \int_{\Gamma_r} d\zeta/J'(\zeta)(z-\zeta)$ *converges uniformly for* $z \in C$.

Proof. Our estimates show that on Γ_r we have $|J'(\zeta)| \geqq c(1 + |y_r|)^d$ because of (3.14) and the fact that $(1 + |y|)^d$ is monotonic in $|y|$. The length of Γ_r is $\leqq c'(1 + |r|)^2$. Moreover, by construction, for $\zeta \in \Gamma_r$ we have $|z-\zeta| \geqq q$. Lemma 3.1 follows immediately if d is sufficiently large because the number of y_r with $|y_r| \leqq |k|$ is by construction less than

$$|k| + j\log(j + |k| + 2).$$

Now, we note that by Cauchy's theorem and the definitions, the integrals $\int_{\Gamma_r} d\zeta/J'(\zeta)(z-\zeta)$ do not depend on z except for the term $1/J(z)$ which depends on whether z lies inside or outside Γ_r. Thus,

$$(3.15) \qquad \sum J'(z) \int_{\Gamma_r} d\zeta/J'(\zeta)(z-\zeta) = \begin{cases} 1 + \sum\limits_r \sum\limits_{z^k \in G_r} J'_k(z) & \text{if } \epsilon \text{ lies in some } \Gamma_r \\ \sum\limits_r \sum\limits_{z^k \in G_r} J'_k(z) & \text{otherwise.} \end{cases}$$

The series on the left of (3.15) obviously converges in the topology of $\boldsymbol{E}'$ (hence, so does the right side), where we have written G_r for all those z^k contained in Γ_r.

Thus we have shown that $\sum\limits_r \sum\limits_{z^k \in G_r} J'_k(z)$ converges in $\boldsymbol{E}'$; as we have noted above (see (3.6) and following) this means we can write

$$(3.16) \qquad \begin{aligned} 1 &= J'(z)K'(z) + \sum_r \sum_{z^k \in G_r} J'_k(z) \\ &= J(z)K(z) + \sum_r \sum_{z^k \in G_r} J'_k(z), \end{aligned}$$

where $K(z) = z^d K'(z)$.

Now, we shall show how to eliminate the restriction that J have only real zeros. By slightly modifying our above constructions we can show that we can construct three sequences of contours:

1. $\{\gamma_k\}$ in a manner similar to $\{\Gamma_k\}$ above
2. $\{\gamma'_k\}$ in the upper half plane
3. $\{\gamma''_k\}$ in the lower half plane

in such a manner that each zero of J is contained in exactly one γ_k, and, if we call $\mathfrak{J}'(z) = z^d J(z)$, $\mathfrak{J}''(z) = \exp(-idz)J'(z)$, $\mathfrak{J}'''(z) = \exp(idz)J'(z)$, then, for d large enough, the three series

$$\Sigma \int_{\gamma_k} d\zeta/\mathfrak{J}'(\zeta)(z-\zeta), \quad \Sigma \int_{\gamma'_k} d\zeta/\mathfrak{J}''(\zeta)(z-\zeta), \quad \Sigma \int_{\gamma''_k} d\zeta/\mathfrak{J}'''(\zeta)(z-\zeta)$$

converge uniformly for $z \in C$. If we denote by G'_r the set of $z^k \in \gamma_r$, G''_r the set of $z^k \in \gamma'_r$, and z''_r the set of $z^k \in \gamma''_r$, then the above shows that the three series

$$\sum_r \sum_{z^k \in \gamma'_r} \mathfrak{J}'(z)\{1/\mathfrak{J}'(z)\}_{z^k}, \quad \sum_r \sum_{z^k \in G''_r} \mathfrak{J}''(z)\{1/\mathfrak{J}''(z)\}_{z^k},$$
$$\sum_r \sum_{z^k \in G'''_r} \mathfrak{J}'''(z)\{1/\mathfrak{J}'''(z)\}_{z^k} \tag{3.17}$$

converge in the topology of E'. It follows immediately from the definitions that

$$1 - \sum_r \sum_{z^k \in G'_r} \mathfrak{J}'(z)\{1/\mathfrak{J}'(z)\}_{z^k} + \sum_r \sum_{z^k \in G''_r} \mathfrak{J}''(z)\{1/\mathfrak{J}''(z)\}_{z^k} + \sum_r \sum_{z^k \in G'''_r} \mathfrak{J}'''(z)\{1/\mathfrak{J}'''(z)\}_{z^k}$$

is a function in E' which vanishes at each z^k to the order j'_k; hence, is of the form $K(z)J(z)$ for some $K \in E'$ (by Theorem 2.6). For each k, moreover, we see that $\mathfrak{J}'_k$, or $\mathfrak{J}''_k$, or $\mathfrak{J}'''_k$ is a multiple of J_k. If we now denote by $\{G_r\}$ some ordering of the three sequences $\{G'_r\}$, $\{G''_r\}$, and $\{G'''_r\}$, then we have:

THEOREM 3.1. *Suppose that $S \in \mathcal{E}'$ is invertible. Then we can find a sequence of distributions $T_k \in \mathcal{E}'$ each of which is of the form $U_k * S_k$ with $U_k \in \mathcal{E}'$ so that for some grouping of terms $\{G_r\}$ the series $\sum_r \sum_{z^k \in G_r} T_k$ converges in the topology of $\mathcal{E}'$. We can find a $W \in \mathcal{E}'$ so that*

$$\delta = S * W + \sum_r \sum_{z^k \in G_r} T_k. \tag{3.18}$$

If $f \in \mathcal{E}$ satisfies $S * f = 0$, then $T_k * f$ are exponential polynomials which depend only on f (not on S or T_k). Hence, the series

$$\sum_r \sum_{z^k \in G_r} T_k * f \tag{3.19}$$

which converges in $\mathcal{E}$ represents the mean periodic expansion of f in terms of exponential polynomials.

Proof. All has been proven except the statement that $S'_k * f$ are exponential polynomials which depend only on f in case $S * f = 0$. First we notice that is is clear that if j''_k denotes the order of the zero of J at z^k

(or d if $z^k=0$) then $(z-z^k)^{j''_k}L_k(z)$ is a multiple of J in the ring $\mathcal{E}'$, where L_k is the Fourier transform T_k. Thus, $(d/dx-z^k)^{j''_k}T_k*f=0$, which means that T_k*f is an exponential polynomial.

We note that T_k*T_l is a multiple of S_k*S_l which is a multiple of S for $k\neq l$. Thus, since the series (3.19) converges to f in the topology of $\mathcal{E}$, we have

$$T_l*f=T_l*T_lf,$$

that is, convolution by T_l is an idempotent for the solutions of $S*f=0$ which is a projection on the exponential polynomial corresponding to z^l.

Thus, if f satisfied an equation $S^1*f=0$ and we have an expansion corresponding to (3.18) for S^1:

$$\delta=S^1*W^1+\sum_r\sum_{k\in G^1_r}T^1{}_k,$$

then we would have by the above

$$f=\sum_r\sum_{k\in G_r}T_k*T^1{}_k*f, \tag{3.20}$$

where only those k appear which are common zeros of J and J^1. By (3.20) we have

$$T_k*f=T^1{}_k*(T_k*f) \tag{3.21}$$

so T_k*f *is* an exponential polynomial for which the degree of the polynomial is $\leqq$ the order of the zero of J^1 at j^k. It is easily seen by Fourier transform that $T^1{}_k$ acts as the identity on such exponential polynomials; we have $T_k*f=T^1{}_k*f$, which shows that the T_k*f depend only on f. (The above assumes that the order of the zero of J at z^k is $\geqq$ order of zero of J^1 at z^k; if this is not the case, the roles of J and J^1 are interchanged.)

This completes the proof of Theorem 3.1.

In case J is not slowly decreasing, then the method of Schwartz (see [27]) shows the existence of a sequence of groupings $\{G_r\}$ such that for each $\epsilon>0$ the series $\sum_r\sum_{k\in G_r}\exp(-\epsilon\,|\,z^k\,|)S_k$ converges in the topology of $\mathcal{E}'$. Moreoever, $\lim_{\epsilon\to 0}\sum_r\sum_{k\in G_r}\exp(-\epsilon\,|\,z^k\,|)S_k$ exists in the topology of $\mathcal{E}'$ and its difference from δ is equal to an element of the *closed* ideal generated by S. This gives Schwartz' main result on the convergence of the mean periodic expansion by means of grouping of terms and Abel convergence factors.

Remark 1. I do not know whether the following weak converse of Theorem 3.1 holds; that is: If there exists an identity like (3.8) with T_k of the form U_k*S_k and if the series on the right side converges, then for

some entire function P of exponential type (but possibly not in $\mathcal{E}'$), $PJ \in \mathcal{E}'$ is slowly decreasing.

Remark 2. Theorem 3.1 can be extended so as to apply to distribution solutions V of $S * V = 0$. In fact, the proof is exactly the same as the proof for $f \in \mathcal{E}$.

4. The intersection of $S * \mathfrak{D}'$ for $S \in \mathcal{E}'$. In this section we shall prove Theorem II of the Introduction, that is, that $\bigcap_{S * \mathcal{E}'} S * \mathfrak{D}' = A$ the space of real analytic functions on R. As we mentioned in the Introduction, the proof that $\bigcap_{S * \mathcal{E}'} S * \mathfrak{D}' \subset A$ depends on the Denjoy-Carleman theorem for quasi-analytic functions which we now recall.

Let $\{M_j\} = M$ be a sequence of positive numbers. We define the class A_M as consisting of all functions f which are defined and indefinitely differentiable on the interval $-1 < x < 1$ and satisfy for some $B, K > 0$

$$(4.1) \qquad |f^{(j)}(x)| \leqq BK^jM^j_j$$

for all x in this interval. The class A_M is called non quasi-analytic if there exists an $f \in A_M$, $f \neq 0$, such that f and all its derivatives vanish at some point x in the interval $-1 \leqq x \leqq 1$.

Theorem of Denjoy-Carleman. A_M is quasi-analytic if and only if the series

$$(4.2) \qquad \sum_{j=0}^{\infty}(1/\bar{M}_j)$$

diverges, where $\bar{M} = \{\bar{M}_j\}$ is the monotonic increasing minorant of M.

From this theorem we deduce the following proposition which will be our key tool in the proof that $\bigcap_{S * \mathcal{E}'} S * \mathfrak{D}' \subset A$:

PROPOSITION 4.1. *Let M be monotonic increasing, with $\sum(1/M_j) < \infty$. Then there exists an $f \in A_M$ which vanishes outside of a compact subset of $-1 \leqq x \leqq 1$ but does not vanish identically.*

Proof. By the Denjoy-Carleman theorem we can find a function $g \in A_M$, $g \not\equiv 0$, which vanishes with all its derivatives at a, say where $-1 < a < 1$. Then g does not vanish identically in at least one of the intervals $-1 < x < a$, $a < x < 1$, we suppose it is the latter. Let b be the greatest lower bound of all $x > a$ for which $g(x) \neq 0$, and set

$$(4.2) \qquad g_1(x) = \begin{cases} g(x) & \text{for } x \geqq b \\ 0 & \text{for } x < b. \end{cases}$$

It is clear that g_1 is again in A_M. Finally, it is clear that for ϵ sufficiently small,

$$f(x) = \begin{cases} g_1(x) g_1(2b - 2x + \epsilon) & \text{for } b \leqq x \leqq b + \epsilon \\ 0 & \text{otherwise} \end{cases} \tag{4.3}$$

vanishes outside of a compact subset of $-1 < x < 1$ and is not identically zero. Since for $b \leqq x \leqq b + 1$ we have

$$| g_1^{(j)}(2b - x + \epsilon) | \leqq BK^j M^j{}_j,$$

all that remains to prove is that whenever two functions satisfy (4.1) so does their product (with possibly different B, K) if M is monotonic.

Let p, q satisfy (4.1). Then for any j,

$$\begin{aligned} |(pq)^{(j)}(x)| &= |\sum_{k=0}^{j} p^{(k)}(x) q^{(j-k)}(x) (C_{k+1}{}^{j+1})| \\ &\leqq \sum_{k=0}^{j} B^2 M^k{}_k K^k M^{j-k}{}_{j-k} K^{j-k} (C_{k+1}{}^{j+1}) \\ &\leqq B^2 M^j{}_j K^j \sum_{k=0}^{j} (C_{k+1}{}^{j+1}) \\ &= B^2 M^j{}_j (2K)^j \end{aligned}$$

because M is monotonic which shows that pq satisfies (4.1) for larger K, B. This completes the proof of Proposition 4.1.

THEOREM 4.2. $\bigcap_{S * \mathcal{E}'} S * \mathcal{D}' \subset A$.

Proof. Let $f \in \bigcap_{S * \mathcal{E}'} S * \mathcal{D}'$ and suppose that f is not real analytic; it is clear anyway that $f \in \mathcal{E}$. Suppose for simplicity that f is not analytic in the neighborhood of $x = 0$. First we note that we can prove easily using the theory of elliptic differential operators and this means we can find a sequence of points $\{a_j\}$ with $|a_j| \leqq 1$ and a corresponding sequence of positive integers m_j which are strictly increasing to infinity so that

$$| f_{m_j}(a_j) | \geqq j^{6nm_j} (2m_j!)^n, \tag{4.4}$$

where we have written f_{m_j} for $\Delta^{m_j} f$, Δ denoting the Laplacian on R.

We shall now define a monotonic sequence $M = \{M_k\}$ as follows:

$$M_k = j^2 p_j \text{ for } k = p_{j-1} + j, p_{j-1} + j + 1, \cdots, p_j + j. \tag{4.5}$$

Here $\{p_j\}$ is a sequence with $p_0 = 1$, $p_j > p_{j-1} + j + 2b_j$ for all $j \geqq 1$, and for all $j \geqq 1$ we have $p_j = 2m_{j'}$ for some j'.

First we note that

$$\sum 1/M_k = \sum (1/j^2) \cdot ([p_j + j - (p_{j-1} + j - 1)]/p_j) < \infty.$$

Thus, by Proposition 4.1 there exists a function $g_1 \in A_M$, support of g_1 contained in $[-1 < x < 1]$. Define the function g on R by $g(x) = g_1(x_1)g_2(x_2), \cdots, g_1(x_n)$. I claim that $f \notin g * \mathfrak{D}'$.

For, suppose $f = g * W$ for some $W \in \mathfrak{D}'$. Since the values of $(g * W)(x)$ for $|x| < 2$ depend only on the values of W on $|x| < 3$, we may assume that W is of the form $\Delta^l h$ for some continuous function h and some l. Then we would have

$$f = g * W = \Delta^l g * h.$$

Moreover, for any r we have

$$\Delta^r f = \Delta^{l+r} g * h. \tag{4.6}$$

We now estimate $\Delta^{l+r_j} g$ by use of the fact that $g_1 \in A_M$. We assume $j > 2l$. First, since $g_1 \in A_M$, we have for any s and any x,

$$\begin{aligned} |(\Delta^s g)(x)| &= |(\partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2)^s g_1(x_1) \cdots g_n(x_n)| \\ &\leqq B^n \sum_{t_1+t_2+\cdots+t_n=2s} |g_1^{(t_1)}(x_1) g_1^{(t_2)}(x_2) \cdots g_1^{(t_n)}(x_n)| \\ &\leqq B^n \sum_{t_1+\cdots+t_2+\cdots+t_n=2s} K^{t_1} M_{t_1}^{t_1} \cdots K^{t_n} M_{t_n}^{t_n} \\ &\leqq B^n n^{2s} K^{2s} M_{2s}^{2s} \end{aligned}$$

because there are n^{2s} terms in the above sum and $\{M_j\}$ is monotonic.

Hence, if $2m_{j'} = p_j$ for some j, we have by (4.5)

$$\begin{aligned} |(\Delta^{l+m_j'} g)(x)| &\leqq B^n (nK)^{2l+2m_j'} M_{2l+2m_j'}{}^{2l+2m_j'} \\ &= B^n (nK)^{2l+2m_j'} (2j^2 m_{j'})^{2l+2m_j'} \\ &= B^n (nKj^2)^{2m_j'} (2m_{j'})^{2m_j'} (nKm_{j'})^{2l}. \end{aligned} \tag{4.7}$$

Now, $(\log x)/x$ is monotonic decreasing for $x \geqq e$, so for $l \geqq 1$ and $m_{j'}$ sufficiently large, $(nKm_{j'})^{2l} \leqq (nK)^{2l}(2l)^{m_j'}$. Thus, (4.7) implies

$$\begin{aligned} |(\Delta^{l+m_j} g)(x)| &\leqq B_1 K_1^{2m_j'} (2m_{j'})^{2m_j'} \\ &\leqq B_2 K_2^{2m_j'} (2m_{j'})! \end{aligned} \tag{4.8}$$

by Stirling's formula, for certain positive constants B_1, B_2, K_1, K_2.

From (4.6) and (4.8) we deduce that for all x with $|x| \leqq 1$ we have

$$|(\Delta^{m_j'} f)(x)| \leqq B_3 K_2^{2m_j'} (2m_{j'})! \tag{4.9}$$

for an infinite number of $m_{j'}$. This clearly contradicts (4.4) and our theorem is proven.

Remark. By a similar kind of argument we can show $(n=1)$ that the intersection of all Carleman non quasi-analytic classes, that is all A_M which are non quasi-analytic, is just A.

We wish now to prove a partial converse of Theorem 4.2. We shall prove also some similar results.

For each $r>0$, denote by A_r the space of functions on R which can be extended to be analytic in the strip C_r defined by $|\mathfrak{I}x|<r$. The topology of A_r is defined by uniform convergence on the compact sets of C_r. This topology is best described by the methods of the theory of infinite derivatives (see [14] for this and following).

By A'_r we denote the dual of A_r; $\hat{A}'_r$ is the Fourier transform of A'_r.

PROPOSITION 4.3. *$\hat{A}'_r$ consists of all entire functions F of exponential type which satisfy*

$$F(z)=O(\exp(l\,|\,\mathfrak{I}z\,|+r'\,|\,\mathfrak{R}z\,|) \tag{4.10}$$

for some l, r' with $r'<r$. The topology of $\hat{A}'_r$ can be described as follows: Let $H(z)$ be and continuous positive function on C with the property that for any l, r' with $r'<r$ we have $\exp(l\,|\,\mathfrak{I}z\,|+r'\,|\,\mathfrak{R}z\,|)=O(H(z))$. Call N_H the sets of $F\in\hat{A}'_r$ which satisfy

$$|F(z)|\leqq H(z) \quad \text{for } z\in C. \tag{4.11}$$

The sets N_H form a fundamental system of neighborhoods of zero in $\hat{A}'_r$.

Proof. For $n=1$ the fact that $\hat{A}'_r$ consists of all entire functions F of exponential type which satisfy (4.10) is a consequence of Polyà's theorem on conjugate diagrams (see e.g. Boas, *Entire Functions*). The statement about the topology can then be proven by combining Polyà's method with my method of describing the topology of $\boldsymbol{H}'$ (see [7]). The passage to $n>1$ presents no difficulties.

A second proof is as follows: Let $\mathcal{E}(C_r)$ be the space of indefinitely differentiable functions on C_r with the usual topology. The topology of the Fourier transform $\boldsymbol{E}'(C_r)$ of $\mathcal{E}'(C_r)$ can be described by the methods of Part III (see [7]). The passage from $\mathcal{E}'(C_r)$ to $\hat{A}'_r$ is accomplished by means of the fundamental principle for systems of constant coefficient equations (see [15], [16]).

Now, let $S\in\mathcal{E}'$. Then $f\to S*f$ is clearly a continuous linear map of $A_r\to A_r$. We claim this map is *onto.* If we make use of the method outlined in the beginning of Section 2, we have to prove that J cannot be too small at infinity, roughly speaking, we have to show that $J(z)>\text{const}\exp(-\epsilon\,|\,z\,|)$

for "enough" $z \in R$. This inequality is obtained by means of subharmonicity (or rather pleurisubharmonicity) of $\log|J|$. More precisely

PROPOSITION 4.4. *We have*

$$\int_R [\log|J(z)|/(1+|z|^2)]dz > -\infty. \tag{4.12}$$

Proof. As in Section 2, we may assume J is bounded on R or even that J is bounded in $\mathfrak{I}z_1 \geqq 0, \cdots, \mathfrak{I}z_n \geqq 0$. By effecting a translation in z and multiplying J by a suitable constant, we may assume that $J(i, i, \cdots, i) = 1$.

We now prove the proposition by induction on n of the stronger proposition:

$$\int_R [\log|J(z_1, \cdots, z_n)|/(1+|z_1|^2)\cdots(1+|z_n|^2)]dz \geqq \log|J(i, i, \cdots, i)|. \tag{4.13}$$

For $n = 1$, inequality (4.13) is a well-known consequence of the subharmonicity of $\log J$. Assume (4.13) is true for values of n smaller than the one in question. Then whenever $\mathfrak{I}z_1 \geqq 0$ we have

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}[\log|J(z_1, z_2, \cdots, z_n)|/(1+|z_2|^2)\cdots (1+|z_n|^2)]dz_2\cdots dz_n \geqq \log|J(z_1, i, \cdots, i)|. \tag{4.14}$$

Now, for fixed $z_2, \cdots, z_n$ the function $z_1 \to \log|J(z_1, z_2, \cdots z_n)|$ is clearly subharmonic and bounded from above. We deduce immediately that the left side of (4.14) is subharmonic and bounded from above for $\mathfrak{I}z_1 \geqq 0$. If we call the left side of (4.14) $\tilde{J}(z_1)$, then the well-known results of subharmonicity show that

$$\int_{-\infty}^{\infty}[\tilde{J}(z_1)/(1+|z_1|^2)]dz_1 \geqq \tilde{J}(i) \geqq \log|J(i, i, \cdots, i)| \tag{4.15}$$

which is the desired result.

We can now obtain our desired estimates for how rapidly J can decrease at infinity on R:

PROPOSITION 4.5. *Given any $\epsilon > 0$, we can find an A so large that for any $x \in R$ with $|x| > A$ there is a $y \in R$ such that*

$$|x - y| < \epsilon|x| \tag{4.16}$$

and

$$|J(y)| \geqq \exp(-\epsilon|x|). \tag{4.17}$$

Proof. Assume there exists an $\epsilon > 0$ with $\epsilon < \frac{1}{2}$ so that (4.16), (4.17) do not hold for a sequence $\{x^j\}$ with $|x^j| > 2|x^{j-1}|$ and $|x^j| \to \infty$. Let R_i be the ring defined as the set of $x \in R$ with $|x^j| \leqq |x| \leqq (1+\epsilon)|x^j|$. By multiplication by a suitable constant we may assume $|J(x)| \leqq 1$ for all $x \in R$.

Then it is immediate that

$$\begin{aligned}
&\int_{R_j} [\log |J(z)|/(1+|z_1|^2)\cdots(1+|z_n|^2)]\,dz \\
(4.18)\qquad &\leqq -\epsilon \int_{R_j} [|z|/(1+|z_1|^2)\cdots(1+|z_n|^2)]\,dz \\
&\leqq -k\epsilon \log(1+\epsilon)
\end{aligned}$$

where $k > 0$ is independent of j. It follows that

$$\Sigma \int_{R_j} [\log |J(z)|/(1+|z_1|^2)\cdots(1+|z_n|^2)]\,dz = -\infty$$

which contradicts Proposition 4.4. This completes the proof of Proposition 4.5.

We can now use the methods of Section 2 (see Lemmas 2.1 and 2.2 and their application in Theorem 2.2) to deduce

THEOREM 4.6. *For any $S \in \mathcal{E}'$ and any $r > 0$, $S * A_r = A_r$.*

In a similar manner we could prove

THEOREM 4.7. *Let $S \in \mathcal{E}'$; let $r > 0$ be fixed. For any $m > 0$ we can find $l, l' > m$ such that if f is analytic in the parallelpiped $|\Re x| < l$, $|\Im x| < r$, then we can find a g which is analytic in $|\Re x| < l'$, $|\Im x| < r$ such that $(S * g)(x) = f(x)$ for $|Rx| < m$, $|\Im x| < r$.*

We wish now to prove the converse of Theorem 4.2, that is, if f is any real analytic function then for any $S \in \mathcal{E}'$ there exists a $g \in \mathcal{E}$ with $S * g = f$. For this purpose, we shall make use of Theorem 4.7 letting m increase to infinity (and $r \to 0$).

Let $S \in \mathcal{E}'$ be fixed; we assume for simplicity of notation that *carrier* $S \subset [-\frac{1}{4} \leqq x \leqq \frac{1}{4}]$. For each m, choose g_m so that g_m is analytic in a parallelpiped and $S * g_m = f$ on $|\Re x| < m+1$, $|\Im x| < r_m$, where r_m is sufficiently small. Now, the functions g_m may not converge on R. However, we can modify g_m by setting $h_m = g_m + k_m$, where $S * k_m = 0$ and k_m is entire. There is now more hope that $\{h_m\}$ will converge.

We have $h_m - h_{m-1} = (g_m - g_{m-1}) = (k_m - k_{m-1})$. Call $k_m - k_{m-1} = l_m$. Then we want to produce l_m so that $S * l_m = 0$ and $(g_m - g_{m-1}) + l_m$ is very

small (and its first m derivatives small) say $\leqq m^{-2}$ on $|\Re x| \leqq m-1$, $\Im x = 0$. We may assume that this inequality holds for smaller values of m. Now, $S*(h_m - h_{m-1}) = 0$ on $|\Re x| \leqq m-1$, $\Im x = 0$. Thus, by a theorem of Malgrange we can choose an exponential polynomial l_m to satisfy the above inequalities. (The result of Malgrange has not been published yet, though it appears in lecture notes. It is closely related to the results of his thesis [19]. I should like to thank Malgrange for pointing this out to me.) The result follows on setting $k_m = k_{m-1} + l_m$.

Now, we can approximate g_m by a polynomial $\tilde{g}_m$ in such a way that $\tilde{g}_m - g_m$ and $S*(\tilde{g}_m - g_m)$ have the property that their first m derivatives are $\leqq m^{-2}$ on $|\Re x| \leqq m$, $\Im x = 0$. This implies that the series

$$\sum [(\tilde{g}_m - \tilde{g}_{m-1}) + l_m]$$

converges in the topology of $\mathcal{E}$, say to g. We have

$$\begin{aligned} S*g &= S*\sum [(\tilde{g}_m - \tilde{g}_{m-1}) + l_m] \\ &= \sum S*[(\tilde{g}_m - \tilde{g}_{m-1}) + l_m] \\ &= \sum S*(\tilde{g}_m - \tilde{g}_{m-1}) \\ &= \lim S*\tilde{g}_m, \end{aligned}$$

where this limit is in the topology of $\mathcal{E}$. Now it is clear that for any $x \in R$,

$$\begin{aligned} \lim (S*\tilde{g}_m)(x) &= \lim (S*g_m)(x) \\ &= f(x). \end{aligned}$$

This, together with Theorem 4.2 gives

THEOREM II. $\bigcap_{S*\mathcal{E}'} S*\mathcal{D}' = \bigcap_{S*\mathcal{E}'} S*\mathcal{E} = A$.

Theorem II is very remarkable. For to show that $\bigcap_{S*\mathcal{E}'} S*\mathcal{D}' \subset$ *real analytic* we used non quasi-analytic classes defined by inequalities of the form $|f^{(r)}(x)| \leqq M_r B^r$. On the other hand, one might suspect that by using non quasi-analytic classes defined by inequalities of the form

$$\left|\sum_{j=1}^{l} f^{(r_j)}(x)\right| \leqq M_{r_1,\cdots,r_l}$$

we could prove a stronger regularity condition on the functions in $\bigcap_{S*\mathcal{E}'} S*\mathcal{D}'$. However, Theorem II shows that no such argument is possible. I do not know if $\bigcap_{S*\mathcal{E}'} S*A = A$. If we would try to prove this by methods similar to our methods of parts I, II, III, or by a method similar to that of Section 2,

we should introduce a "natural" topology in A. Then we should try to describe the Fourier transform of $\hat{A}'$ in a method like that described in the beginning of Section 2. However, it seems extremely unlikely that this is possible for reasons given below:

One possible way of putting a topology on A is to consider A as the union of all spaces of functions analytic in a fixed open set containing R (these space being given the usual compact-open topology) and then giving A the inductive limit topology (see e.g. [14] or [30]). Then we see easily from Proposition 4.3 and the definition of an inductive limit that $\hat{A}'$ consists of all entire functions F of exponential type which satisfy for some l

$$F(z) = O(\exp(l \mid \mathfrak{I} z \mid + \epsilon \mid \mathfrak{R} z \mid)$$

for all $\epsilon > 0$.

Suppose that the topology of $\hat{A}'$ could be described by means of positive continuous functions $\{H\}$ with the property that a fundamental system of neighborhoods of zero in $\hat{A}'$ consists of those sets N for which there is an H so that N consists of all $F \in \hat{A}'$ which satisfy $|F(z)| \leqq H(z)$ for all $z \in C$. Now, clearly, for any $F \in \hat{A}'$ and any H there is an $a > 0$ so that $a \mid F(z) \mid \leqq H(z)$ for all $z \in C$. By considering products of entire functions of exponential type zero (constructed by power series) and exponentials we deuce that for each $l > 0$ there is an $\epsilon_l > 0$ so that

$$\exp(l \mid \mathfrak{I} z \mid + \epsilon_l \mid \mathfrak{R} z \mid) = O(H(z)). \tag{4.19}$$

We shall denote by $\hat{B}'$ the space of functions of $\hat{A}'$ with the topology defined by all functions H which satisfy (4.19).

I claim that inequality (4.19) implies that there exists an ϵ independent of l so that

$$\exp(l \mid \mathfrak{I} z \mid + \epsilon \mid \mathfrak{R} z \mid) = O(H(z)) \tag{4.20}$$

for all l. This implies immediately that $\hat{B}' \neq \hat{A}'$ and, in fact that a function of B must be analytic in a strip about R. In fact, using Proposition 4.3 we could show that B is the inductive limit of the spaces A_r.

For simplicity I assume $n = 1$. We may clearly assume that the ϵ_l decrease with increasing l. Then for $l > 1$, from the relationships

$$\exp(\mid \mathfrak{I} z \mid + \epsilon_1 \mid \mathfrak{R} z \mid) = O(H(z))$$
$$\exp(l \mid \mathfrak{I} z \mid + \epsilon_l \mid \mathfrak{R} z \mid) = O(H(z)).$$

I want to conclude that

$$\exp(l' \mid \mathfrak{I} z \mid + \tfrac{1}{4}\epsilon_1 \mid \mathfrak{R} z \mid) = O(H(z)),$$

where $l' \to \infty$ with l. Thus I want to show that

$$\begin{aligned}(4.21)\quad &\exp(l' \mid \mathfrak{I}z \mid + \tfrac{1}{4}\epsilon_1 \mid \mathfrak{R}z \mid) \\ &\qquad \leqq \text{const.} \max\{\exp(\mid \mathfrak{I}z \mid + \epsilon_1 \mid \mathfrak{R}z \mid), \exp(l \mid \mathfrak{I}z \mid + \epsilon_l \mid \mathfrak{R}z \mid)\}.\end{aligned}$$

To prove (4.21) we may suppose for simplicity that we are in the quadrant $C^{(1)}$: $\mathfrak{I}z \geqq 0$, $\mathfrak{R}z \geqq 0$. Then let $C_1^{(1)}$ be the subset where $l\mathfrak{I}z \geqq \frac{1}{2}\epsilon_1 \mathfrak{R}z$ and let $C_2^{(1)}$ be the complement. In $C_1^{(1)}$ we have

$$\exp(\tfrac{1}{2}l\mathfrak{I}z + \tfrac{1}{4}\epsilon_1 \mathfrak{R}z) \leqq \exp(l\mathfrak{I}z).$$

In $C_2^{(1)}$ we have

$$\exp(\tfrac{1}{2}l\mathfrak{I}z + \tfrac{1}{4}\epsilon_1 \mathfrak{R}z) \leqq \exp(\epsilon_1 \mathfrak{R}z).$$

Thus (4.21) is proven with $l' = \frac{1}{2}l$ is the desired result.

There is another possible method of introducing a topology on A: By our above, A is the projective limit of the spaces $\mathcal{E}_M$, that is, intersection which are Carleman non quasi-analytic. (Here $\mathcal{E}_M$ is the space of functions in E which satisfy inequalities of the form (4.1) on every compact set; $\mathcal{E}_M$ is given a natural topology as in [14].) We could give A the projective limit topology, that is, a convex set $N \subset A$ is a neighborhood of zero if it is the intersection with A of a neighborhood of zero in some E_M. Call K the set of functions of A with this topology. Then we define, as usual, the Fourier transform $\hat{K}'$ of the dual K' of K. Assume the topology of $\hat{K}'$ can be described by functions $\{H\}$ as above for $\hat{B}'$. Then I want to show that these functions H also satisfy (4.20) for some ϵ independent of l.

To prove my assertion, I know (see e.g. [14]) that the set $\{a_j M_j^{-j} z^j\}$ is bounded in B' whenever $a_j = O(\epsilon^j)$ for all $\epsilon > 0$. (For the linear functions $f \to i^j a_j M_j^{-j} f^{(j)}(0)$ form a bounded set in E_M and $a_j M_j^{-j} z^j$ are their Fourier transforms.) Suppose for example that for no $\epsilon > 0$ is $\exp(\epsilon \mid \mathfrak{R}z \mid) = O(H(z))$. Then there exists an infinite sequence of points $\{z_k\}$ with $\{\mid \mathfrak{R}z_k \mid\}$, $\{\mid z_k \mid\}$ sufficiently lacunary (see below) such that $H(z_k) < \exp(1/k^3 \mid \mathfrak{R}z_k \mid)$.

I want to construct $\{M_j\}$ so that for a suitable x_k, we have

$$(4.22)\qquad k^{-j} M_j^{-j} \mid z_k \mid^j = \exp|k^{-3} \mid \mathfrak{R}z_k \mid).$$

For this we need

$$(4.23)\qquad M_j = k^{-1} \mid z_k \mid exp(-j^{-1} k^{-3} \mid \mathfrak{R}z_k \mid).$$

For given k, choose $j = [\mid \mathfrak{R}z_k \mid k^{-3}] + 1$ and for this choice of j define M_j by (4.23). We assume $\{\mid \mathfrak{R}z_k \mid\}$ is lacunary enough so that

$$j(k) = [\mid \mathfrak{R}z_k \mid k^{-3}] + 1$$

is a strictly increasing function of k. The definition of $\{M_j\}$ is completed by setting $M_{j'} = M_j$ whenever $j' < j$, j of the form $[| \mathcal{R} z_k | k^{-3}] + 1$ and for no other j'' with $j' < j'' < j$ is j'' of the form $[| \mathcal{R} z_k | k^{-3}] +$.

I have only to show that, with this choice of $\{M_j\}$, $\sum M_j^{-1} < \infty$. We have

$$\begin{aligned}\sum M_j^{-1} &\leqq \sum_k k \mid z_k \mid^{-1} \cdot (j(k) - j(k-1))\\ &\leqq \sum_k k \mid z_k \mid^{-1} j(k)\\ &\leqq \sum_k k \mid z_k \mid^{-1}\{[| \mathcal{R} z_k | k^{-3}] + 1\}\\ &\leqq \sum k \mid z_k \mid^{-1} + \sum k^{-2}\\ &< \infty\end{aligned}$$

if we assume, as we may, that $| z_k | \geqq k^3$.

This completes the proof of our assertion that for some ϵ_0 we must have $\exp(\epsilon_0 | \mathcal{R} z |) = O(H(z))$. (The above shows even that $\exp(\epsilon_0 | z |) = O(H(z))$.) By considering products of functions of the form $a_j M_j^{-j} z_j^k$ with exponentials we may deduce by the above method that, for any H as above and any $l > 0$, we can find an ϵ_l so that $\exp(\epsilon_l | \mathcal{R} z | + l | \mathcal{I} z |) = O(H(z))$. Thus, as in the example of $\hat{B}'$ above, the functions H are not sufficient to define the topology of $\hat{K}'$.

Remark. I do not know if the topologies of K and A are the same.

5. Elliptic operators. In this section we shall consider C^∞ and entire elliptic operators $S \in \mathcal{E}'$, and we shall characterize them completely. In order to explain the principles which underline our theory, we shall first give a heuristic argument in case $n = 1$.

Call J the Fourier transform of S; suppose for simplicity that S is invertible. Then we know by the results of Section 3 that each distribution $V \in \mathcal{D}'$ which satisfies $S * V = 0$ can be expanded in a convergent series of exponential polynomial solutions; these latter correspond to the zeros of $J(z)$. When must such a convergent series be C^∞ (entire)?

Let a, b be real numbers and consider $\exp(iax + bx)$. Its derivative is $(ia + b)\exp(iax + bx)$. We have

$$|(d/dx)\exp(iax + bx)| = | ai + b | \exp(bx). \tag{5.1}$$

The above (5.1) shows that for any $l > 0$,

$$\begin{aligned}\max_{|x| \leqq l} |(d/dx)\exp(iax + bx)| &\leqq \max_{|x| \leqq l + |\log|ai+b||/|b|} \exp(bx)\\ &= \max_{|x| \leqq l + |\log|ai+b||/|b|} \exp(iax + bx).\end{aligned} \tag{5.2}$$

From (5.2) it follows that if we have a series $\sum C_j \exp(ia_jx + b_jx)$ which converges absolutely and uniformly on every compact set, say to $f(x)$, then we can obtain bounds for f' on an interval $|x| \leqq l$ in terms of bounds for f on some interval $|x| \leqq l'$ if we know that we can find an $M > 0$ so that, for all j,

$$(5.3) \qquad |\,log\,|\,a_ji + b_j\,||/|\,b_j\,| \leqq M.$$

Hence, by repeating this argument and noting that any finite number of terms of the series $\sum c_j \exp(ia_jx + b_jx)$ do not affect the question of whether $f \in \mathcal{E}$, we see that a sufficient condition to know that $f \in \mathcal{E}$ is

$$(5.4) \qquad \limsup |\log|\,a_ji + b_j\,||/|\,b_j\,| < \infty.$$

By a similar argument, we would know that f is entire if

$$(5.5) \qquad \limsup |\,a_j\,|/|\,b_j\,| < \infty.$$

We are therefore led to guess that (5.4) (where $a_j - ib_j$ are the zeros of J) is the condition that S be weakly C^∞ elliptic, and (5.5) is the condition that S be entire elliptic. We shall see that this is actually the case.

The proof of one half of the above, namely, if S does not satisfy condition (5.4), or (5.5), then S is not weakly C^∞ elliptic, or entire elliptic in x_1, will be accomplished by means of exponential sums, that is, e.g., if S does not satisfy (5.5), then we can find a series $\sum c_j \exp(iz^j\cdot)$, where $J(z^j) = 0$, which converges in the space $\mathcal{E}$ to a function which is not analytic in x_1.

In the theory of elliptic differential equations there are in the literature essentially two different methods to obtain results of the type of Theorem III. The first method depends essentially on Gärding's lemma (see [17]) which in turn depends on the fact that J is large at infinity on R. Since, as we shall see later, there exist $S \in \mathcal{E}$ which are C^∞ elliptic but are small at infinity, it does not seem that such a method can be of use in our case.

The second method depends upon the construction of an elementary solution for S which is a C^∞ (or analytic) function outside of a neighborhood of the origin. This method cannot hope to succeed in case S is not invertible for then by Theorem 2.2 there exists no elementary solution which is a distribution in the sense of Schwartz. In case S is invertible, this method does work, and it is outlined in my note [10], and will be presented in detail below. However, even in case S is not invertible, there is hope to find an elementary solution with desired properties which is not a distribution in the sense of Schwartz but which is a continuous linear function on a suitable space of non quasi-analytic functions. However, as Theorem 6.2 shows, this method cannot work for arbitrary S.

There is a third possibility for proving our results. Let us reexamine the case $n = 1$. It would be fairly easy to obtain our above heuristic argument with the methods of Section 3 to obtain the desired result in case S is invertible. Even in case S is not invertible by combining these methods with certain properties of Abel limits, there seems to be hops to prove an extension of our results, but we shall not discuss this here. This method is fairly close to my fundamental principle and will be discussed elsewhere (see [16]).

Before proving our assertions about ellipticity, I wish to state several preliminary propositions on distributions which are C^∞ in x_1:

PROPOSITION 5.1. *Let $T \in D'$ be C^∞ in $(x_1, \cdots, x_r)$; then for $h(x_{r+1}, \cdots, x_n)$ an indefinitely function of compact support, we have $T * h \in \mathcal{E}$, that is, T is C^∞ in $(x_1, \cdots, x_r)$ relative to $(x_{r+1}, \cdots, x_n)$.*

(Here by $T * h$ we mean the convolution of T with the direct product (see [24]) of h with the δ distribution in $(x_1, \cdots, x_r)$.)

Proof. We prove the result in case $r = 1$ as the general proof is similar. For any $j_1, j_2, \cdots, j_n$, we have

$$(5.6)\qquad \begin{aligned} &(\partial^{j_1+\cdots+j_n}/\partial x_1^{j_1}\cdots\partial x_n^{j_n})(T * h) \\ &\qquad = (\partial^{j_1}/\partial x_1^{j_1})T * (\partial^{j_2+\cdots+j_n}/\partial x_2^{j_2}\cdots\partial x_n^{j_n})h. \end{aligned}$$

Now, let K be any compact set in R. Then because h is of compact support, we can find a compact set $L \subset R$ so that the values of $T * h$ on K depend only on the values of T on L. By definition, the distributions $(\partial^{j_1}/\partial x_1^{j_1})T$ are of bounded order on L. Thus, the right side of (5.6) is of bounded order on K (this bound is independent of $j_1, j_2, \cdots, j_n$). Hence, al lthe derivatives of $T * h$ are of bounded order on K, which proves (see [24]) that $T * h$ is an indefinitely differentiable function on K. Since K was arbitrary, it follows that $T * h \in \mathcal{E}$. Thus Proposition 5.1 is proven.

Note that a similar argument shows that if T is entire in x_1, then $T * h$ is entire in x_1.

For T a distribution which is C^∞ in $(x_1, \cdots, x_r)$ relative to $(x_{r+1}, \cdots, x_n)$ we define the restriction $T_{x_1=0,\cdots,x_r=0}$ of T to the plane $x_1 = x_2 = \cdots = x_r = 0$ by

$$(5.7)\qquad T_{x_1=0,\cdots,x_r=0} \cdot h = (T * h)(0)$$

for any $h \in \mathcal{D}(x_{r+1}, \cdots, x_n)$. We prove that this restriction is a distribution. More generally, we have

PROPOSITION 5.2. *If T is C^∞ in $(x_1, \cdots, x_r)$* relative to *$(x_{r+1}, \cdots, x_n)$,*

then $h \to T * h$ is a continuous map of $\mathfrak{D}(x_{r+1}, \cdots, x_n)$ into $\mathcal{E}$. A necessary and sufficient condition for $W \in D'$ to be C^∞ in $(x_1, \cdots, x_r)$ relative to $(x_{r+1}, \cdots, x_n)$ is that the orders of $\{(\partial^j/\partial x_1^{j_1} \cdots \partial x_r^{j_r})W\}$ should be zero in $x_1, \cdots, x_r$, that is, for any $j_1, \cdots, j_r$ and for $K \subset R$ compact, we can find a differential operator D in $x_{r+1}, \cdots, x_n$ and a measure μ on K so that

$$(\partial^j/\partial x_1^{j_1} \cdots \partial x_r^{j_r})W = D\mu. \tag{5.8}$$

Proof. Let K' be a fixed compact set in $(x_{r+1}, \cdots, x_n)$. Then $h \to T * h$ is a linear map of $\mathfrak{D}_{K'}(x_{r+1}, \cdots, x_n) \to \mathcal{E}$. Moreover, this map is closed, for if $h \to h'$, then $T * h \to T * h'$ in the topology of $\mathfrak{D}'$, so if $T * h$ converges in the topology of $\mathcal{E}$, it can only converge to $T * h'$. Thus by the closed graph theorem, $h \to T * h$ is continuous on each $\mathfrak{D}_{K'}(x_{r+1}, \cdots, x_n)$. Hence, the map is continuous on $\mathfrak{D}(x_{r+1}, \cdots, x_n)$ by the definition of an inductive limit.

Next, if W satisfies the condition stated, then arguing as in the proof of Proposition 5.1 we see that W is C^∞ in $(x_1, \cdots, x_r)$ relative to $(x_{r+1}, \cdots, x_n)$. Conversely, let W be C^∞ in $(x_1, \cdots, x_r)$ relative to $(x_{r+1}, \cdots, x_n)$. For simplicity of notation we assume $n = 2$, $r = 1$; the general case is treated similarly. Suppose there is a cube K such that the derivatives $\{(\partial^j/\partial x_1^j)W\}$ are not of zero order in x_1 on K. Then there exists a j and sequence of functions f^k with supports contained in K so that

$$\max |(\partial^j/\partial x_1^j)(\partial^l/\partial x_2^l)(f^k)(x)| \leqq 1 \quad \text{for } l = 0, 1, \cdots, j$$

but

$$|W \cdot (\partial^j/\partial x_1^j)f^k| \geqq k.$$

Now, the set $\{(\partial^j/\partial x_1^j)f^k_{x_1=a}\}$ is clearly bounded in $\mathfrak{D}$. ($f^k_{x_1=a}$ is the function $x_2 \to f^{k1}(a, x_2)$.) Thus, by the first part of Proposition 5.2, the set $\{W * (\partial^j/\partial x_1^j)f^k_{x_1=a}\}$ is bounded in $\mathcal{E}$. Since $\{a\}$ is compact and since $(\partial^j/\partial x_1^j)f^k_{x_1=a}$ and hence $W * (\partial^j/\partial x_1^j)f^k_{x_1=a}$ depend continuously on a, it follows that $\{\int W * (\partial^j/\partial x_1^j)f^k_{x_1=a}\, da\}$ is bounded. But,

$$|\int W * (\partial^j/\partial x_1^j)f^k_{x_1=a}\, da| = |W \cdot (\partial^j/\partial x_1^j)f^k| \to \infty$$

which completes the proof of Proposition 5.2.

PROPOSITION 5.3. *Suppose J does not satisfy* (1.1) *and* (1.2), *that is, there exists a sequence of points $\{{}_jz\}$ and a $k > 0$ such that $J({}_jz) = 0$, $|{}_jz| \to \infty$, $|{}_jz_1|^k \geqq |{}_jz|$ for j large enough, but*

$$\limsup |\mathfrak{J}({}_jz)|/\log(1 + |{}_jz_1|) = M < \infty.$$

Then S is not C^∞ elliptic in x_1; in fact, there exists a $T \in \mathfrak{D}'$ with $S * T = 0$, and T not C^∞ in x_1. Given any q, we may even choose T to be a q-times differentiable function on $|x| \leqq q$.

Proof. I shall assume that the sequence $\{|z^j|\}$ is strictly increasing to infinity and is, in fact, sufficiently lacunary to satisfy the conditions below (all this may be assured by taking a suitable subsequence). We may also assume for simplicity that $\mathfrak{J}({}_jz_a) \geqq 0$ for all j, a. We can choose the ${}_jz$ so lacunary that $|{}_{j+1}z| \geqq |{}_jz| + j$. Call $W = \sum \delta_{{}_jz}$; it is clear from our explicit expression for the topology of $\boldsymbol{D}$ that the series $\sum \delta_{{}_jz}$ converges in $\boldsymbol{D}'$. Thus, the Fourier transform T of W lies in $\mathfrak{D}'$ and satisfies $S * T = 0$.

I claim that for no sequence $\{c_l\}$ of positive numbers is $\{c_l(\partial^l/\partial x_1{}^l)T\}$ bounded in $\mathfrak{D}'$; that is, T is not C^∞ in x_1. For, assume that for some $\{c^l\}$ it is true that $B = \{c_l(\partial^l/\partial x_1{}^l)T\}$ is bounded in $\mathfrak{D}'$. Then it follows from the definition of the topology of $\mathfrak{D}'$ that we can find an r so large that B is bounded on the bounded sets of $\mathfrak{D}_1{}^r$ (r-times differentiable functions with supports in $|x| \leqq 1$ with Schwartz [24] topology). I shall show that this is impossible.

Using the methods of the last part of the proof of Theorem 2.2 we can construct a sequence of function $\{F_t\} \subset \mathfrak{D}_1$ such that

1. $F_t(0) = 1$,
2. $|F_t(z)| \leqq 1$ for $\mathfrak{J}z_1 \leqq 0, \mathfrak{J}z_2 \leqq 0, \cdots, \mathfrak{J}z_n \leqq 0$,
3. $|F_t(z)| \leqq (t/z)^t \exp|\mathfrak{J}z|$ for all z.

Define

$$G_t(z) = F_t(z - z_t)/|{}_tz|^{r+2}. \tag{5.9}$$

Then we see immediately that

4. $\{G_t\}$ is bounded in $\mathfrak{D}_1{}^r$,
5. For any s we can find t_0 so large that

$$|z_1{}^sW \cdot G_t| = |\sum_j {}_jz_1{}^sG_t({}_jz)| \geqq \tfrac{1}{2}|{}_tz_1{}^s|/|{}_tz|^{r+2} \tag{5.10}$$

for $t \geqq t_0$. Hence, if s is large enough, it follows that $\{|z_1{}^sW \cdot G_t|\}$ is not bounded.

We have proved that for any r we can find an s large enough so that $z_1{}^sW$ is not bounded on the bounded sets of $\mathfrak{D}_1{}^r$, hence, $(\partial^s/\partial x_1{}^s)T$ is not bounded on the bounded sets of $\mathfrak{D}_1{}^r$. Hence, T is not C^∞ in x_1 so our result is established.

By considering sums of the form $\sum |{}_jz|^{-p}\delta_{{}_jz}$ we would, given q, produce a distribution T which satisfies $S * T = 0$, is C^q on the set $|x| \leqq q$ and is not C^∞ in x_1 by taking p large enough. (However, we cannot, in general, produce a T which is a distribution of finite order as follows from results below.)

By reasoning as in Proposition 5.3 we can establish

PROPOSITION 5.4. *Suppose there exists a sequence of points* $\{_jz\}$ *and a* $k > 0$ *such that* $J(_jz) = 0$, $|_jz| \to \infty$, $|_jz_1|^k \geqq |_jz|$ *for j large enough, but*

$$\limsup |\mathfrak{I}(_jz)|/\log(1 + |_jz_1|) = 0. \tag{5.11}$$

Then S is not weakly C^∞ *elliptic in* x_1. *Given any q, we can find a function f which is q times differentiable and satisfies* $S * f = 0$ *but which is not* C^∞ *in* x_1 *(even considered as an element of* $\mathfrak{D}'_F$*).*

We now consider the analog for analytic elliptic:

PROPOSITION 5.5. *Suppose there exists a sequence of points* $\{_jz\}$ *and a* $k > 0$ *such that* $J(_jz) = 0$, $|_jz_1| \to \infty$, $|_jz_1| > k \log |_jz|$ *for j large enough, but*

$$\limsup |\mathfrak{I}(_jz)|/|_jz_1| = 0. \tag{5.12}$$

Then J is not entire elliptic in x_1. *We can find a function* $f \in \mathcal{E}$ *with* $S * f = 0$ *such that f is not entire in* x_1.

Proof. We may clearly assume that $|_{j+1}z| \geqq |_jz| + j$, and, as in the proof of Proposition 5.1, that $\mathfrak{I}(_jz_a) \geqq 0$ for all j, a. Now the series $\sum \delta_{_jz}$ may not converge in $\mathfrak{D}'$ if $\mathfrak{I}(_jz)$ is too large. However, if $\{b_j\}$ is any sequence of positive numbers increasing to infinity, then the series

$$\sum \exp(-b_j |\mathfrak{I}(_jz)|)\delta_{_jz}$$

converges in $\mathfrak{D}'$ as is readily verified. We shall choose

$$b_j = |_jz_1|^{\frac{1}{2}}(|\mathfrak{I}(_jz)| + 1)^{-\frac{1}{2}}. \tag{5.13}$$

By our hypothesis, $b_j \to \infty$. Next, let $c_j \to \infty$ at a very slow rate (to be specified later). It is readily verified that the series

$$W = \sum |_jz|^{-c_j} \exp(-b_j |\mathfrak{I}(_jz)|)\delta_{_jz} \tag{5.14}$$

converges in the topology of $\mathcal{E}$. If f denotes the Fourier transform of W, then $f \in \mathcal{E}$ satisfies $S * f = 0$. I claim that if the $|_jz|$ are lacunary enough (to be explained later), then f is not entire in x_1, even if f is considered as an element of D'.

If f were entire in x_1, then for r large enough

$$B = \{(1/s!)(\partial^s/\partial x_1^s)f\}$$

would be bounded on the bounded sets of $\mathfrak{D}_1^r$. We shall show that this is impossible if $c_j \to \infty$ slowly enough and if the numbers $|_jz| \to \infty$ fast enough.

Let us note the following: If $s = [|z|/2]$, then

$$
\begin{aligned}
|z|^s/s! &\geqq \exp([|z|/2]\log|z| - [|z|/2]\log[|z|/2]) \\
&\geqq \exp(2[|z|/2]). \\
&\geqq e^{-1}\exp(|z|).
\end{aligned}
\tag{5.15}
$$

We assume the points ${}_jz$ are so lacunary that $|{}_{j+1}z| \geqq e\,|{}_jz|$. We now return to the notation of the proof of Proposition 5.3. The functions G_t satisfy 4. Moreover, if $s = [|{}_tz_1|/2]$, then by (5.15) and the definition of b_t, we have

$$
\begin{aligned}
|(1/s!)\,{}_tz_1{}^s|\exp(-b_t|\mathfrak{I}({}_tz)|) &\geqq \exp(-1 + |{}_tz_1| - |{}_tz_1|^{\frac{1}{2}}|\mathfrak{I}({}_tz)|^{\frac{1}{2}}) \\
&\geqq \exp(\tfrac{1}{2}|{}_tz_1|)
\end{aligned}
\tag{5.16}
$$

for t large enough because of (5.12). Thus, making use of the properties of the G_t, we have, for this s if t is large enough,

$$
\begin{aligned}
&|(1/s!)z_1{}^sW\cdot G_t| \\
&\quad = |\sum_j (1/s!)\,{}_jz_1{}^sG_t({}_jz)\,|{}_jz|^{-c_j-r-2}\exp(-b_j|\mathfrak{I}({}_jz)|)| \\
&\quad \geqq \tfrac{1}{2}\exp(\tfrac{1}{2}|{}_tz_1|)\,|{}_tz|^{-c_j-r-2}.
\end{aligned}
\tag{5.17}
$$

We can clearly choose $\{c_j\}$ with $c_j \to \infty$ so that for any r the right hand side is unbounded. This completes the proof of Proposition 5.5.

Remark. The same method can be used to show that f is not real analytic in x_1 in the neighborhood of any point.

We are now ready to prove the converses of these three propositions.

Let $T \in \mathfrak{D}'$; we say T is C^∞ in x_1 on an open set $\Omega \subset R$ if we can find positive numbers b_j so that $\{b_j(\partial^j/\partial x_1{}^j)T\}$ is a bounded set of distributions on Ω. (A similar definition for T entire in x_1 on Ω.) Let $S \in \mathcal{E}'$; we say that $P \in \mathfrak{D}'$ is a *C^∞ in x_1 parametrix for S* if P is C^∞ in x_1 outside of some neighborhood of the origin and

$$S * P = \delta + W, \tag{5.18}$$

where W is C^∞ in x_1. We say that P is a *C^∞ in x_1 parametrix of finite order* if W, $P \in \mathfrak{D}'_F$ and if we can find an m so that for each $r > 0$ we can find an e_r so that the first r derivatives of P with respect to x_1 are distributions of order $\leqq m$ for $|x| > e_r$. We say that P is an *entire in x_1 parametrix* for S if W is analytic in x_1 for $|\mathfrak{I}x_1| < r$ and if given any $r > 0$ we can find a d_r so that P is analytic in x_1 for $|\mathfrak{I}x_1| < r$ in $|x| > d_r$.

The importance of the parametrix comes from

PROPOSITION 5.6. (a) *If $S \in \mathcal{E}'$ has a C^∞ in x_1 parametrix, then S is C^∞ elliptic in x_1.*

(b) *If $S \in \mathcal{E}'$ has a C^∞ in x_1 paramerix of finite order, then S is weakly C^∞ elliptic in x_1.*

(c) *If $S \in \mathcal{E}'$ has an entire in x_1 parametrix, then S is entire elliptic in x_1.*

Proof. All the proofs are similar, so we prove (c) for illustration. Let $S * T$ be entire in x_1; we have to show that T is entire in x_1. We show that, in the neighborhood of the origin $|x| < m$, T is analytic in x_1 for $|\mathfrak{I}x_1| < r$. Let $h \in D$ be 1 for $|x| < r$. Let $h \in D$ be 1 for $|x| < m'$ (to be chosen later). Then consider hT. We have by (5.18)

$$\begin{aligned} hT &= hT * \delta \\ &= hT * P * S - hT * W. \end{aligned} \tag{5.19}$$

Now, $hT * W$ is certainly entire in x_1 since W is. Moreover, $h * P * S = (S * hT) * P$. We know that if m' is large enough, $S * hT = S * T$ on $|x| < m''$ (where m'' can be made arbitrarily large). Now, the restriction of $(S * hT) * P$ to $|x| < m$ depends on the convolution of P with the restriction of $S * hT$ to the set $|x| < m'$ (m' large enough) and on the convolution of hT with the restriction of P to $|x| > m'''$ (which again can be made arbitrarily large). Thus, our assertion is established.

PROPOSITION 5.7. *Let $S \in \mathcal{E}'$ be invertible for $\mathcal{D}'$ and satisfy* (1.1) *and* (1.2). *Then there exists a C^∞ in x_1 parametrix for S.*

Proof. In order to make the proof clear, we shall give the proof first in case $n = 1$. Then there are only a finite number of real zeros of J, and the modulus of the imaginary parts of the zeros $\to \infty$ faster than any multiple of the log of the modulus of the real part, We want to define something like

$$\int_{-\infty}^{\infty} [\exp(ixz)/J(z)]\, dx. \tag{5.20}$$

Let K be chosen so large that all the zeros z_j of J satisfy $|\mathfrak{R} z_j| < K - 1$. Then instead of the above integral, we shall consider

$$\int_{-\infty}^{-K} [\exp(ixz)/J(z)]\, dz + \int_{K}^{\infty} [\exp(ixz)/J(z)]\, dz \tag{5.21}$$

which differs from (5.20) by an integral over a compact set. We shall see that the integral over the compact set does not matter.

Next, we consider $\int_K^\infty [\exp(ixz)/J(z)]dz$. Assume x is large and positive. Then it would be natural to try to shift the contour as high as we can in the complex z plane, i.e., make $\mathfrak{I}z$ as large as possible, and in this way make use of the smallness of $\exp(ixz)$. This can be done because J has no zeros unless $\mathfrak{I}z$ is very large.

The only trouble we encounter is that $\int_K^\infty [\exp(ixz)/J(z)]dz$ does not have to have a meaning in the usual sense because J may $\to\infty$ at infinity. Therefore we modify thet argument slightly as follows: Let $f \in \mathfrak{D}$; then instead of (5.21) we consider

$$(5.22) \qquad \int_{-\infty}^{-K} [F(z)/J(z)]dz + \int_K^\infty [F(z)/J(z)]dz.$$

We shall show that each integral on the right of (5.22) exists in the usual sense (because J is slowly decreasing) and if f vanishes for $x < a$, where $a > 0$, will be prescribed later, then we can shift the contour as we described above. We then define $P \in \mathfrak{D}'$ by

$$(5.23) \qquad P\cdot g = \int_{-\infty}^{-K} [G(z)/J(z)]dz + \int_K^\infty [G(z)/J(z)]dz$$

for any $g \in D$. We verify easily that P is a parametrix for S. Moreover, the above argument that P is C^∞ for $x > a$ (similarly, for $x < -a$). Thus our result will be proven.

Now we proceed with the details of the proof: Since J is slowly decreasing, there exists an $A > 0$ so that for each $z \in R$ we can find a $w \in R$ ($w = w(z)$) with $|w - z| < A\log(1 + |z|)$ and $|J(w)| \geqq (A + |z|)^{-A}$. If $|z|$ is large enough (and z_0 real), then the circle $|z - z_0| \leqq 2A\log(1 + |z_0|)$ will contain no zeros of J. Now by the minimum modulus theorem (see [8]) we can draw about $w_0 = w(z_0)$ a circle in the complex z plane of radius between $(3/2)A\log(1 + |z_0|)$ and $2A\log(1 + |z_0|)$ on which $|J(z)| \geqq (A' + |z_0|)^{-A'}$ for some A'. But $1/J$ is analytic in this circle. Thus we also have $|J(z_0)| \geqq (A' + |z_0|)^{-A'}$ by the maximum modulus theorem. Hence, for any $F \in \boldsymbol{D}$ the right side of (5.23) exists as an absolutely convergent integral and P defined by (5.23) is a distribution (of finite order).

Next we want to shift the contour in $\int_K^\infty [F(z)/J(z)]dz$ if f vanishes for $x < a$. We pick a sequence of numbers K_j with $K_0 = K$, $K_{j+1} \geqq K_j + 1$ and so that for z' real, $z' \geqq K_j$, $J(z)$ does not vanish if $|z - z'| \leqq 2j\log(1 + |z'|)$.

We define a curve Γ by: Γ consists of all z for which $\mathfrak{I}z = j\log Rz$ if

$K_j \leqq Rz \leqq K_{j+1}$, and Γ is made continuous by joining the various parts by vertical lines. It is trivial to verify that the total length of Γ lying above the interval $[K_0, b]$ is $<$ const. b^2. Moreover, we can again apply our minimum modulus-maximum modulus argument as above to deduce that

$$(5.24) \qquad |J(z)| \geqq (A'' + |z|)^{-A''} \exp(-A'' |\mathfrak{J} z|)$$

for $z \in \Gamma$. On the other hand, f vanishes for $x \leqq a$; a simple argument shows that this implies that

$$(5.25) \qquad |F(z)| \leqq A'''(A'' + |z|)^{-A''-4} \exp(-(A''+1)|\mathfrak{J} z|)$$

if a is large enough. Moreover, if $f \in B$, where $B \subset \mathfrak{D}$ is a bounded set of functions which vanish for $x < a$, then we may assume that inequality (5.25) holds for all $f \in B$ for the same A'''.

All the above shows us that $\int_\Gamma [F(z)/J(z)]dz$ exists in the absolute sense. Moreover, again using the fact that f vanishes for $x < a$, we deduce immediately that

$$(5.26) \qquad \int_K^\infty [F(z)/J(z)]dz = \int_\Gamma [F(z)/J(z)]dz.$$

A similar construction holds for $\int_{-\infty}^{-K} [F(z)/J(z)]dz$.

We now want to show that P is C^∞ for $x > a$. (A similar method works for $x < -a$.) Let B be a bounded set in $\mathfrak{D}$ of functions which vanish for $x > a$. Then for $f \in B$ and for any r we deduce as above

$$(5.27) \qquad \int_K^\infty z^r [F(z)/J(z)]dz = \int_\Gamma z^r [F(z)/J(z)]dz.$$

By the construction of Γ, we have on Γ,

$$(5.28) \qquad |z^r| \exp(-|\mathfrak{J} z|) \leqq b_r.$$

Combining this with (5.24) and (5.25) we deduce that for all $f \in B$,

$$(5.29) \qquad \left| \int_K^\infty z^r [F(z)/J(z)]dz \right| \leqq b_r A''''.$$

Here b_r depends only on r and J (but is indepenednt of B) and A'''' depends only on B.

A similar construction for $\int_{-\infty}^{-K}$ shows that

$$(5.30) \qquad |P \cdot f^{(r)}| \leqq 2A'''' b_r$$

for all $f \in B$. This means that $\{b_r^{-1}P^{(r)}\}$ is bounded on every bounded set of functions in $\mathfrak{D}$ which vanish for $x < a$, so that $\{b_r^{-1}P^{(r)}\}$ is bounded in $\mathfrak{D}'$ for $x > a$. This means that P is C^∞ for $x > a$; a similar method applies for $x < -a$.

Finally, we must prove that P is a parametrix for S, that is, $S * P = \delta + W$, where W is a C^∞ function. We have for $f \in \mathfrak{D}$,

$$\begin{aligned} S * P \cdot f &= P \cdot S * f \\ &= \int_{-\infty}^{-K} [J(z)F(z)/J(z)]\,dz + \int_{K}^{\infty} [J(z)F(z)/J(z)]\,dz \\ &= \int_{-\infty}^{-K} + \int_{K}^{\infty} F(z)\,dz \\ &= \int_{-\infty}^{\infty} F(z)\,dz - \int_{-K}^{K} F(z)\,dz \\ &= \delta \cdot f + W \cdot f, \end{aligned}$$

where $W(x) = \int_{-1}^{1} \exp(ixz)\,dz$ is an entire function of exponential type. Thus our result is proven in case $n = 1$.

We show now how to modify the above argument in case $n \neq 1$. Our first task is to define P. Since J is slowly decreasing, by Lemma 2.2 each $z \in R$ can be surrounded by a set on which $|J| \geqq (A + |z|)^{-A}$. Moreover, the maximum distance from z to any point on this set is $\leqq A \log(1 + |z|)$. Call R' the set of $z \in R$ for which $|J(z)| \geqq (A + |z|)^{-A}$, and set $R'' = R - R'$. Then we define P by

$$P \cdot f = \int_{R'} [F(z)/J(z)]\,dz.$$

Then P clearly is a distribution.

Let $z \in R''$, then the above construction shows there must be a point $w \in V$ with $|w - z| < A \log(1 + |z|)$. In particular, $|\mathfrak{I}w| < A \log(1 + |z|)$ and even

$$|\mathfrak{I}w| < A' \log(1 + |w|). \tag{5.31}$$

Thus, we know from the fact that J satisfies (1.1) and (1.2) that for all such z,

$$\liminf \log|w_1|/\log|w| = 0. \tag{5.32}$$

It follows from (5.32) and the fact that $|z - w| < A \log(1 + |z|)$ that we also have

$$\liminf_{z \in R'',\, |z| \to \infty} \log|z_1|/\log|z| = 0. \tag{5.33}$$

This inequality will be used to show that P is a C^∞ in z_1 parametrix for S.

Our next task is to shift the contour from R' to show that P is C^∞ in x_1 outside of a neighborhood of the origin. We show first that P is C^∞ in x_1 in the half-space $x_n > a$ for a sufficiently large. A similar argument applies to $x_n < -a$ and also to the half-spaces $x_j > a$, $x_j < -a$ so that we shall know that P is C^∞ in x_1 for $|x| > a$.

Let $f \in \mathcal{D}$ vanish for $x_n \leqq a$. Let $z_1, \cdots, z_{n-1}$ be fixed. Then we shall define a new contour $\Gamma(z_1, \cdots, z_{n-1})$ in such a way that

$$(5.34) \qquad P \cdot f = \int \cdots \int dz_1 \cdots dz_{n-1} \int_{\Gamma(z_1, \cdots, z_{n-1})} [F(z)/J(z)] dz_n.$$

Call $J_{z_1, \cdots, z_{n-1}}$ the function $z \to J(z_1, \cdots, z_{n-1}, z_n)$. For fixed $z_1, \cdots, z_{n-1}$ we divide the real z_n line into subsets $K_j(z_1, \cdots, z_{n-1})$. $K_j(z_1, \cdots, z_{n-1})$ consists of all real z_n with the property that $J_{z_1, \cdots, z_{n-1}}(z'_n)$ does not vanish for $|z'_n - z_n| \leqq 2j \log(1 + |z_n|)$.

Now we define the $\Gamma(z_1, \cdots, z_{n-1})$ by: For each integer l, $\Gamma(z_1, \cdots, z_{-n1})$ consists (above $[l \leqq t \leqq l+1]$) of those points z_n with $\mathfrak{I} z_n = j \log(1 + |Rz_n|)$, $l \leqq Rz_n \leqq l+1$, if j is the smallest integer such that all points in $[l \leqq t \leqq l+1]$ belong to $R'_{z_1, \cdots, z_{n-1}} \cap K_j(z_1, \cdots, z_{n-1})$. If there is a t in $[l \leqq t \leqq l+1]$ for which $t \notin R'_{z_1, \cdots, z_{n+1}}$, then $\Gamma(z_1, \cdots, z_{n-1})$ above $[l \leqq t \leqq l+1]$ is just $R'_{z_1, \cdots, z_{n-1}} \cap [l \leqq t \leqq l+1]$. Finally, $\Gamma(z_1, \cdots, z_{n-1})$ is completed by joining the various pieces by vertical lines.

It is clear from the definitions that the lenth of $\Gamma(z_1, \cdots, z_{n-1})$ for $|z_n| \leqq b$ is $<$ const. b^2. Thus, for any $f \in \mathcal{D}$ which vanishes for $x_n > a$ we have

$$(5.35) \qquad \int \cdots \int dz_1 \cdots dz_{n-1} \int_{\Gamma(z_1, \cdots, z_{n-1})} [F(z)/J(z)] dz$$

$$= \int_{R'} [F(z)/J(z)] dz$$

$$= P \cdot f$$

as in the case $n = 1$.

To prove that P is C^∞ in x_1 outside a neighborhood of 0, we have to show that we can find constants b_r so that for all real $z_1, \cdots, z_{n-1}$ and all $z_n \in \Gamma(z_1, \cdots, z_{n-1})$ we have

$$(5.36) \qquad |z_1{}^r \exp(-|\mathfrak{I}(z_n)|) \leqq b_r(1 + |z|).$$

Let us consider $z_1{}^r \exp(-|\mathfrak{I} z_n|)$ for $z_n \in \Gamma(z_1, \cdots, z_{n-1})$. Since J satisfies (1.1) and (1.2), we can find a $c_r > 0$ so large that for all $z' \in R$, if $|z'| > c_r$, then no point of the circle $|z''_n - z'_n| < 2r \log(|z'| + 1) + 1$

can contain a zero of $J_{z'_1,\cdots,z'_n}(z''_n)$ unless z' belongs to a set R''' on which $|z'_1|^r < d_r(1+|z|)$. In particular, this means that (for r large enough) the imaginary part of any point in $\Gamma(z_1,\cdots,z_{n-1})$ will be $> r\log|z'|$ except for $z' \in R'''$ or for $|z'| \leqq c_r$. This means that

$$(5.37)\qquad |z_1|^r \exp(-|\mathfrak{I}z|) \leqq d_r(1+|z|) + c_r^r$$

which implies inequality (5.36).

We can now proceed exactly as in the case $n=1$ to show that P is C^∞ in x_1 for $x_n > a$. Similar results hold for $x_n < -a$ and $x_j > a$ or $x_j < -a$, that is, P is C^∞ in x_1 for $|x| > a$.

Finally, P is a C^∞ in x_1 parametrix for S. For, we have for any $f \in D$,

$$\begin{aligned} S * P\cdot f = P\cdot S * f &= \int_{R'} [J(z)F(z)/J(z)]\,dz \\ &= \int_R F(z)\,dz - \int_{R''} F(z)\,dz \\ &= \delta\cdot f - \int_{R''} F(z)\,dz. \end{aligned}$$

If we set $W\cdot f = \int_{R''} F(z)\,dz$, then W is clearly a distribution. Moreover, by similar calculations as above, inequality (5.33) implies that W is C^∞ in x_1. This completes the proof of Proposition 5.7.

Putting the above Propositions 5.6, 5.7, 5.3 together we have (see Theorem III of the Introduction)

THEOREM 5.8. *Let $S \in \mathcal{E}'$ be invertible for $\mathcal{D}'$. Then* (1.1) *and* (1.2) *are a set of necessary and sufficient conditions for S to be C^∞ elliptic in x_1.*

A similar method applies to show

THEOREM 5.9. *Let $S \in \mathcal{E}'$ be invertible for $\mathcal{D}'_F$. Then S is weakly C^∞ elliptic in x_1 if and only if we can find an $m > 0$ so that for each $r > 0$ we can find a $b_r > 0$ with the property that*

$$(5.38)\qquad |\mathfrak{I}z| \geqq m\log(1+|z_1|)$$

whenever $z \in V$ and

$$(5.39)\qquad |z_1^r| \geqq b_r(1+|z|).$$

THEOREM 5.10. *Let $S \in \mathcal{E}'$ be invertible for $\mathcal{D}'$. Then S is entire elliptic in x_1 if and only if we can find an $m > 0$ so that*

$$(5.40)\qquad |\mathfrak{I}z| \geqq m(1+|z_1|)$$

whenever $z \in V$ *and*

$$|z_1| \geqq m^{-1} \log(1 + |z|). \tag{5.41}$$

We can also use similar methods to prove the analogs for relative ellipticity:

THEOREM 5.11. *Let* $S \in \mathcal{E}'$ *be invertible for* $\mathfrak{D}'$. *Then* S *is* C^∞ *elliptic in* $(x_1, \cdots, x_l)$ *relative to* $(x_{l+1}, \cdots, x_n)$ *if and only if for each* $r \geqq 0$ *we can find a* $b_r > 0$ *with the property that*

$$|\mathfrak{I}z| \geqq r \log(1 + |(1 + |(z_1, \cdots, z_l)|) \tag{5.42}$$

whenever $z \in V$ *and*

$$|(z_1, \cdots, z_l)|^r \geqq b_r(1 + |(z_{l+1}, \cdots, z_n)|)^{b_r}(1 + |z|). \tag{5.43}$$

THEOREM 5.12. *Let* $S \in \mathcal{E}'$ *be invertible for* $\mathfrak{D}'_F$. *Then* S *is weakly* C^∞ *elliptic in* $(x_1, \cdots, x_l)$ *relative to* $(x_{l+1}, \cdots, x_n)$ *if and only if we can find an* $m > 0$ *so that for each* $r > 0$ *we can find a* $b_r > 0$ *with the property that*

$$|\mathfrak{I}z| \geqq m \log(1 + |(z_1, \cdots, z_l)|) \tag{5.44}$$

whenever $z \in V$ *and*

$$|(z_1, \cdots, z_l)|^r \geqq b_r(1 + |(z_{l+1}, \cdots, z_n)|)^{b_r}(1 + |z|). \tag{5.45}$$

COROLLARY 5.12. *Let* $S \in \mathcal{E}'$ *be invertible for* $\mathfrak{D}'$. *Suppose that in* x_1, S *is a differential operator with leading coefficient* 1. *Then* S *is weakly* C^∞ *elliptic in* x_1 *relative to* $(x_2, \cdots, x_n)$. *If* ∂ *is a partial differential operator in all variables and* $x_1 = 0$ *is non characteristic for* ∂, *then* ∂ *is* C^∞ *elliptic in* x_1 *relative to* $(x_2, \cdots, x_n)$.

Proof. Our hypotheses imply $J = z_1^p + \sum J_j z_1^j$, where $J_j \in \mathbf{E}'(z_2, \cdots, z_n)$. Then for fixed $(z_2, \cdots, z_n)$, $J(z)$ does not vanish if

$$|z_1| \geqq \text{const.} \max |J_j(z_2, \cdots, z_n)|.$$

Now, for some $A > 0$

$$\max |J_j(z_2, \cdots, z_n)| \leqq A(1 + |(z_2, \cdots, z_n)|)^A \exp(A\, |\mathfrak{I}(z_2, \cdots, z_n)|).$$

Thus, $J(z)$ does not vanish if

$$|z_1| \geqq \text{const}(1 + |z_2, \cdots, z_n|)^A \exp(A\, |\mathfrak{I}(z_2, \cdots, z_n)|).$$

By Theorem 5.12 this implies that J is weakly C^∞ ellipitic in x_1 relative to $(x_2, \cdots, x_n)$.

The proof for ∂ is similar.

THEOREM 5.13. *Let S in $\mathcal{E}'$ be invertible for $\mathcal{D}'$. Then S is entire elliptic in $(x_1,\cdots,x_l)$ relative to $(x_{l+1},\cdots,x_n)$ if and only if we can find an $m>0$ so that*

$$(5.46) \qquad |\mathfrak{I}z| \geqq m(1+|(z_1,\cdots,z_l)|)$$

whenever $z\in V$ and

$$(5.47) \qquad |(z_1,\cdots,z_l)| \geqq m^{-1}\log(1+|z|).$$

That is, relative entire ellipticity is the same as entire ellipticity.

Remark. We could have defined relative ellpiticity in terms of convolution by elements of Carleman non quasi-analytic classes instead of $\mathcal{D}$. In this case, the right hand side of (5.47) would be changed and the class of relative entire elliptic operators would presumably be different for some Carleman classes, though I have not constructed any examples.

In case S is invertible, we can show, using the methods of Section 2, that every distribution W which is C^∞ (entire) in x_1 can be written in the form $S*T$, where T is again C^∞ (entire) in x_1. Thus, if S is invertible, then every solution T of the equation $S*T=W$, where W is C^∞ (entire) in x_1, is also C^∞ (entire) in x_1 provided we know that every solution T of $S*T=0$ is C^∞ (entire) in x_1. Thus, if S is invertible, then the conditions for ellipticity can be stated in terms of the homogeneous equation $S*T=0$.

I do not know if there exists an S which is not invertible and is C^∞ elliptic in x_1. However, in case we are considering C^∞ (entire) ellipticity in all variables, we shall see that no S with the corresponding property can exist without being invertible, that is, if for all $f\in\mathcal{E}$ $(f\in\mathcal{H})$ all the distribution solutions T of

$$S*T=f$$

are again in $\mathcal{E}$ (in $\mathcal{H}$), then S is invertible.

Definition. J is called *extra slowly decreasing in z_1* if for each $a>0$ and each $k>0$ there exists an $m>0$ so large that

$$(5.48) \qquad \liminf_{|\mathfrak{I}(z)|\leqq a+a\log(1+|\mathcal{R}z|)} |z_1|^m\,|J(z)| = \infty.$$

THEOREM 5.14. *If J is not extra slowly decreasing in z_1, then there exists a $T\in\mathcal{D}'$ so that T is not C^∞ in x_1 but $S*T$ is C^∞ in x_1. In particular, if S is C^∞ elliptic in all variables, then S is invertible.*

Proof. Assume J is not extra slowly decreasing in z_1. Then we can find a $k>0$ and a sequence $\{_jz\}$ with $|\mathfrak{I}(_jz)| \leqq a + a\log(1+|\mathfrak{R}(_jz)|)$, $|_jz| \to \infty$, $|_jz_1|^k > |_jz|$ and for each m,

$$\limsup |_jz_1|^m |J(_jz)| = M_m < \infty. \tag{5.49}$$

As in the proof of Proposition 5.3, the series $\sum \delta_{jz}$ converges in $\mathfrak{D}'$ to W whose Fourier transform T is not C^∞ in x_1. But using

$$JW = \sum J(_jz)\delta_{jz} \tag{5.50}$$

we see easily that $\{z_1{}^m(M_m)^{-1}JW\}$ is bounded on the bounded sets of $\mathfrak{D}$, hence, is bounded in $\mathfrak{D}'$. Thus, $S*T$ is C^∞ in x_1 which proves our assertion.

Remark 1. We could easily make T to be a q times differentiable function on the set $|x| \leqq q$.

Remark 2. The above theorem, as well as the succeeding ones, can be easily extended to the cases where "distribution" is replaced by "distribution of finite order," and "C^∞ in x_1" is replaced by "entire in x_1."

From Theorems 5.14 and 5.8 we deduce immediately

THEOREM 5.15. *A necessary and sufficient condition that S be C^∞ elliptic in all variables is:*

$$\liminf_{z\in V,\,|z|\to\infty} |\mathfrak{I}z|/\log(1+|z|) = \infty. \tag{5.51}$$

Remark. It seems that the conditions: $S*T=0$ implies T is C^∞ in x_1, and J is extra slowly decreasing in z_1 should imply that S is C^∞ elliptic in x_1. However, I do not know how to decide this.

I should now like to give several examples:

Example 1. We give an example of an S which is entire elliptic in x_1 but is not C^∞ elliptic in all variables. A trivial example is $S=\partial/\partial x_1$. A less trivial examples is $S=\partial/\partial_1 - i\partial/\partial x_2 - i\partial/\partial x_3$. Then $J(z)=iz_1+iz_2+z_3$. Write $z_j=\xi_j+i\eta_j$. Then $J(z)=0$ is equivalent to

$$\xi_1 = -\eta_2-\eta_3$$

$$\eta_1=\xi_2+\xi_3.$$

Now, $|\mathfrak{I}z| = |\eta_1|+|\eta_2|+|\eta_3|$, so that

$$|\mathfrak{I}z/z_1| = |\eta_1/z_1| + (|\eta_2|+|\eta_3|)/|z_1|. \tag{5.32}$$

If $|\eta_1| \leqq \frac{1}{2}|z_1|$, then $|\xi_1| > \frac{1}{2}|z_1|$. Thus

$$(5.53) \qquad (|\eta_2| + |\eta_3|)/|z_1| \geqq |\eta_2 + \eta_3|/|z_1| = |\xi_1|/|z_1| > \tfrac{1}{2}.$$

This combined with (5.52) shows that for all $z \in V$, $|\mathfrak{J}z|/|z_1| \geqq \frac{1}{2}$. Hence, by Theorem 5.10 S is entire elliptic in x_1.

Since $V \cap R$ is not compact, S is clearly not C^∞ elliptic in all variables.

Example 2. I want to construct an example of an $S \in \mathfrak{D}$ which satisfies (1.1) and (1.2). Since $S \in \mathfrak{D}$, S cannot be invertible and S cannot be C^∞ ellipitic.

Let us consider first the case $n = 1$. We can construct an $F \in \mathfrak{D}$ which is even, $F(0) = 1$, $F(x) = O(\exp(-|x|^{\frac{3}{2}})$; the possibility of constructing such an F is well-known from the theory of quasi-analytic functions. We write

$$(5.54) \qquad F(z) = \pi(1 - z^2/a_j^2).$$

We may assume, by replacing a_j by $|a_j|$ if necessary, that the a_j are real and positive, because if we replace a_j by $|a_j|$, then for any real x, we have

$$\begin{aligned} |1 - x^2/|a_j|^2| &= |a_j|^{-2}\,||a_j|^2 - x^2| \\ &\leqq |a_j|^{-2}\,|a_j^2 - x^2| \\ &= |1 - x^2/a_j^2|. \end{aligned}$$

Thus, the infinite product (5.54) does not increase for $z \in R$ when we replace a_j by $|a_j|$.

Next we define

$$(5.55) \qquad J(z) = \pi(1 - z^2/(a_j^2 + ia_j^{\frac{5}{4}}).$$

For j large, the imaginary part of $(a_j^2 + ia_j^{\frac{5}{4}})^{\frac{1}{2}}$ is about $a_j^{\frac{1}{4}}$. Thus it remains to show that $J \in \boldsymbol{D}$.

Actually, it is very difficult to show that $J \in \boldsymbol{D}$ by comparing it with F directly. However, it follows easily from the minimum modulus theorem that

$$(5.56) \qquad F((x^2 + i|x|^{\frac{3}{2}})^{\frac{1}{2}}) = O(\exp(-\tfrac{1}{2}|x|^{\frac{1}{2}}).$$

(Since F is even, it does not matter which square root we take on the left.) There is much more hope in showing $J(x)/F((x^2 + i|x|^{\frac{3}{2}})^{\frac{1}{2}})$ is bounded and so to conclude that $J \in \boldsymbol{D}$.

We have for $x > 0$,

$$(5.57) \qquad \begin{aligned} J(x)/F((x^2 + ix^{\frac{3}{4}})^2) &= \Pi(1 - x^2/(a_j^2 + ia_j^{\frac{5}{4}}))/\Pi(1 - (x^2 + ix^{\frac{3}{2}}/a_j^2)) \\ &= \Pi((a_j^2 + a_j^{\frac{5}{4}} - x^2)/(a_j^2 - (x^2 + ix^{\frac{3}{2}}))) \cdot \Pi(a_j^2/(a_j^2 + ia_j^{\frac{5}{4}})). \end{aligned}$$

The reciprocal of the last product is

$$\prod((a_j^2 + i a_j^{\frac{5}{4}})/a_j^2) = \prod(1 + i a_j^{-\frac{3}{4}}).$$

It is well known (see e.g. [29]) that this latter product converges absolutely. Hence, so does $\prod(a_j^2/(a_j^2 + i a_j^{\frac{5}{4}}))$.

We are thus left to consider the product

$$\prod (a_j^2 + i a_j^{\frac{5}{4}} - x^2)/(a_j^2 - x^2 - i x^{\frac{3}{2}}).$$

For $x^{\frac{3}{2}} \geqq a_j^{\frac{5}{4}}$, the terms of the product are all of modulus $\leqq 1$. If $x^{\frac{3}{2}} < a_j^{\frac{5}{4}}$ and if $a_j > a_0$ (independent of x), then

$$\begin{aligned} |(a_j^2 + i a_j^{\frac{5}{4}} - x^2)/(a_j^2 - x^2 - i x^{\frac{3}{2}})|^2 &= ((a_j^2 - x^2)^2 + a_j^{\frac{5}{2}})/((a_j^2 - x^2)^2 + x^3) \\ &(a_j^2 - x^2)^2/((a_j^2 - x^2)^2 + x^3) + a_j^{\frac{5}{2}}/((a_j^2 - x^2)^2 + x^3) \\ &\leqq 1 + 2a_j^{-\frac{3}{2}}. \end{aligned}$$

Hence, except possibly for a polynomial factor in x,

$$\prod |(a_j^2 + i a_j^{\frac{5}{4}} - x^2)/(a_j^2 - x^2 - i x^{\frac{3}{2}})| \leqq M,$$

where M is independent of x. This proves that $J \in \boldsymbol{D}$, which is the desired result.

In case $n > 1$, we define J_1 as follows:

$$J_1(z) = J((z_1^2 + z_2^2 + \cdots + z_n^2)^{\frac{1}{2}}), \tag{5.58}$$

where J is defined as in (5.55). Note that since J is even, J_1 is entire. It is immediately verified that $J_1 \in \boldsymbol{D}$. We want to examine the zeros of J_1. If z is such a zero, then for some j,

$$z_1^2 + z_2^2 + \cdots + z_n^2 = a_j^2 + i a_j. \tag{5.59}$$

We write $z_k = \xi_k + \eta_k$. Then (5.59) becomes

$$\begin{gathered} \xi_1^2 + \xi_2^2 + \cdots + \xi_n^2 - \eta_1^2 - \eta_2^2 - \cdots - \eta_n^2 = a_j^2 \\ 2\xi_1\eta_1 + 2\xi_2\eta_2 + \cdots + 2\eta_n\xi_n = a_j^{\frac{5}{4}} \end{gathered} \tag{5.60}$$

or

$$\begin{gathered} |\xi|^2 - |\eta|^2 = a_j^2 \\ 2|\xi \cdot \eta| = a_j^{\frac{5}{4}} \end{gathered} \tag{5.61}$$

Now, if $|\xi| > 2|\eta|$, then (5.61) shows that $|\xi|^2 - \frac{1}{4}|\xi|^2 \leqq a_j^2$, so that $|\xi| \leqq 2a_j$. Hence, by (5.61) again,

$$a_j^{\frac{5}{4}} \leqq 2|\xi| \cdot |\eta| \leqq 4a_j |\eta|.$$

Thus, $|\eta| \geqq \frac{1}{4} a_j^{\frac{1}{4}}$, which is the desired result.

Remark. I suspect that all distribution solutions of $S * T = 0$ are in $\mathcal{E}$, but I cannot prove this. In case $n = 1$, it may be possible to prove this by use of Schwartz' mean-periodic expansion.

Example 3. We give an example of an $S \in \mathcal{E}'$ such that all distribution solutions of $S * W = 0$ are in C^∞ but S is not invertible, hence, S is not C^∞ elliptic: Let $n = 1$, and let S_1 be C^∞ elliptic and let J_1 have infinitely many zeros; we may assume J is even. Choose a very lacunary infinite sequence of the zeros a_j of J_1, so lacunary that $J_2(z) = \prod (1 - z^2/a_j^2)$ is of zero order, or even that if $M_2(r) = \max_{|z|=r} |J_2(z)|$, then $\log M_2(r) = O((\log r)^2)$. Then (see Boas, *Entire Functions*, p. 50, the proof of Theorem 3.6.1), if $m_2(r)$ denotes the minimum of $J_2(z)$ on a circle of radius r, we will have $\log m_2(r) \geqq \frac{1}{2} \log M_2(r)$ except on a sequence of intervals $I_j = \{b_j \leqq r \leqq c_j\}$, where each I_j contains some $|a_j|$ and where $\sum_{c_j \leqq R} (c_j - b_j) \leqq \frac{1}{2} R$ for R sufficiently large.

Suppose we choose the a_j in such a way that $|a_{j+1}| > (10 + |a_j|)^2$ for all j. For j large enough, if we call $x_j = |a_j|/3$, then for any $y \in R$ with $|y - x_j| < |x_j|/6$ we have

$$|J_2(y)| \geqq m_2(|y|) \geqq \tfrac{1}{2} M_2(|y|).$$

On the other hand, by Liouville's theorem, for all p, $\liminf M_2(r)/r^p = \infty$. It follows that $J(z) = J_1(z)/J_2(z)$ cannot be slowly decreasing. This can be seen fairly easy by applying the minimum modulus as above to the functions $J_2(z)/(1 - z^2/a_j^2)$ from which it is deduced that except for the effect of the factors $(1 - z^2/a_j^2)$ near a_j, the division of J_1 by J_2 serves to decrease $|J_1|$. The term $1 - z^2/a_j^2$ is handled by using the fact that for $|z| \geqq 4$ we have $|1 - z^2/a_j^2| \geqq |z|^{-2}$ except in circles of radius 4 about $\pm a_j$; these circles can be treated by the maximum modulus theorem. This argument shows that, in fact, $J \in \mathbf{D}$.

Next we note that J_2 is not in $\mathbf{E}'$. However, using the methods of [14] we can find a space B of functions of compact support such that the inverse Fourier transform S_2 of J_2 is well defined on B.

Now, suppose $T \in \mathcal{D}_1$ satisfies $S * T = 0$. We consider T as an element of B' and it must therefore satisfy, as an element of B',

$$0 = S_2 * S * T = S_1 * T.$$

Since B is dense in $\mathcal{D}$ this implies $S_1 * T = 0$ as an element of $\mathcal{D}'$. Hence, $T \in \mathcal{E}$ because S_1 is C^∞ elliptic. This proves our assertion.

For $n > 1$ we could proceed as in example 2 above.

Example 4. I give an example of an $S \in \mathcal{E}'$ which satisfies (1.1) and (1.2) in all variables but for which there exists a distribution T satisfying $S * T = 0$, with $T \notin \mathcal{E}$. We assume $n = 1$; the passage to $n > 1$ is as in Example 2 above.

Let us define $J_1(z) = \prod \cos(z/j!\, j \log^2 j)^{j!}$. Let us choose $J_2 \in \boldsymbol{D}$ so that $J_2(x)\exp(|x|^{2/3})$ is bounded on R. (The existence of J_2 is guaranteed by the Denjoy-Carlemann theorem on quasi-analytic classes.) Let $J_3(z) = J_1(z)J_2(z)$. As in Example 2 above, we can "shift" the zeros of J_3 to the curve $\mathfrak{I}z = |Rz|^{\frac{1}{2}}$ and obtain a function $J \in \boldsymbol{D}$.

J has a zero at $j!\, j \log^2 j + i(j!\, j)^{\frac{1}{2}} \log j$ of order $j!$. Thus, if s denotes the Fourier transform of J, then we have solutions of the equation $S * f = 0$ of the form

$$f_j(x) = Q_j(x)\exp(izj!\, j \log^2 j - x(j!\, j)^{\frac{1}{2}}(\log j)^{\frac{1}{2}}),$$

where Q_j is any polynomial of degree $\leqq j!$. Of course, I want to choose Q_j in a suitable manner. If all the solutions of $S * T = 0$ were in E, then we could find a $b > 0$ so large that

$$f'_j(0) \leqq b \max_{|x| \leqq b} f_j(x). \tag{5.62}$$

This is a consequence of the closed graph theorem which shows that δ' (the derivative of the δ) is continuous on the subspace of $\mathcal{E}$ of solutions of S in the topology induced by the space of continuous functions on R.

Thus, I want to choose Q_j to violate (5.62). For this purpose I am led to the following problem: Let P_k be a polynomial of degree k such that $|P_k(x)| \exp g_k |x| \leqq 1$ for $|x| \leqq 1$. Suppose $p_k(0) = 0$ and $|P'_k(0)| \geqq 1$; then how large can g_k be? We assert that we can take $g_k \geqq \operatorname{const} k^{\frac{1}{2}}$.

Before proving this, I show how it can be used to prove our result: Inequality (5.62) shows that we can find a $b > 0$ so that for any polynomial Q_j of degree $\leqq j!$ with $Q_j(0) = 0$ we have

$$\begin{aligned} &|Q'_j(0)|\,|ij!\, j \log^2 j - (j!\, j)^{\frac{1}{2}}(\log j)^{\frac{1}{2}}| \\ &\qquad \leqq b \max_{|x| \leqq b} |Q_j(x)| \exp((j!\, j)^{\frac{1}{2}}(\log j)^{\frac{1}{2}}|x|). \end{aligned} \tag{5.63}$$

I claim we cannot have $b = 1$ and the general case follows by the simple transformation $x \to x/b$.

Our above construction of the P_k shows we can choose Q_j so that

$$\max_{|x| \leqq 1} |Q_j(x)| \exp((j!)^{\frac{1}{2}}|x|) \leqq 1$$

but $|Q'_j(0)| \geqq 1$. That is, the right side of (5.63) is $\leqq 1$ but the left side is arbitrarily large. This is a contradiction and proves our result.

Remark. We could actually construct a $T \in \mathcal{D}'$ which satisfies $S * T = 0$, but $T \notin \mathcal{E}$; we could even make T differentiable as often as we want.

It remains to prove our assertion on the existence of P_k: We pick $P_{2k+1} = x(1-x^2)^k$. Then clearly $P_k(0) = 0$, $P'_k(0) = 1$. If we could show that $\max\limits_{0 \leqq x \leqq 1} (1-x^2)^k e^{\frac{1}{2}x}$ is bounded from above uniformly in k, then clearly so is $\max\limits_{|x| \leqq 1} | P_{2k+1}(x) | \, e^{k^{\frac{1}{2}}|x|}$.

The derivative of $(1-x^2)^k e^{k^{\frac{1}{2}}x}$ is

$$k^{\frac{1}{2}}(1-k^2)^k e^{k^{\frac{1}{2}}x} - 2kx(1-x^2)^{k-1} e^{k^{\frac{1}{2}}x}.$$

This vanishes (for $k > 1$) if $x = \pm 1$ or if

$$1 - x^2 - 2k^{\frac{1}{2}}x = 0,$$

$$x^2 + 2k^{\frac{1}{2}}x - 1 = 0$$

$$x = -k^{\frac{1}{2}} \pm (k+1)^{\frac{1}{2}}.$$

The value we are interested is thus $x = -k^{\frac{1}{2}} + (k+1)^{\frac{1}{2}}$. Since $(1-x^2)^k e^{k^{\frac{1}{2}}x}$ vanishes at $x = 0, 1$, is non-negative in the interval $[0,1]$ and has $x = -k^{\frac{1}{2}} + (k+1)^{\frac{1}{2}}$ as its only critical point in this interval, it follows that $x = -k^{\frac{1}{2}} + (k+1)^{\frac{1}{2}}$ is a maximum. The value of the function at this point is

$$(1 - ((k+1)^{\frac{1}{2}} - k^{\frac{1}{2}})^2)^k e^{k^{\frac{1}{2}}((k+1)^{\frac{1}{2}} - (k^{\frac{1}{2}}))}.$$

Note that $(k+1)^{\frac{1}{2}} - k^{\frac{1}{2}} < \frac{1}{2}$ for, squaring, we have to show that

$$k + 1 < \tfrac{1}{4} + k + k^{\frac{1}{2}}$$

which is clear. Moreover, $(k+1)^{\frac{1}{2}} - k^{\frac{1}{2}} > \frac{1}{4}k^{-\frac{1}{2}}$ for squaring we must show that $k + 1 > k + \frac{1}{16}k^{-1} + \frac{1}{2}$ which is again clear. Thus,

$$\begin{aligned} &(1 - ((k+1)^{\frac{1}{2}} - k^{\frac{1}{2}})^2)^k e^{k^{\frac{1}{2}}((k+1)^{\frac{1}{2}} - k^{\frac{1}{2}})} \\ &\qquad \leqq (1 - \tfrac{1}{16}k^{-1})^k e^{\frac{1}{2}k^{\frac{1}{2}}}. \end{aligned} \tag{5.64}$$

The right side of (5.64) behaves for large k like $\exp(-\frac{1}{16}k + \frac{1}{2}k^{\frac{1}{2}}) \to 0$. Thus the left side of (5.64) is uniformly bounded in k which completes the proof of our assertion.

Remark. It would be of interest to find the best possible g_k and also the corresponding P_k.

I wish now to show that those S which are entire elliptic in all variables are just (essentially) the classical elliptic differential operators. More generally, we have

THEOREM 5.16. *Let S be entire elliptic in x_1. Then, in x_1, S is the composition of a differential operator with a translation, that is, we can find a real number a and a finite sequence $\{S_j\}$ of distributions which are independent of x_1 so that*

$$S = \sum S_j \times (\partial^j/\partial x_1{}^j * \tau_a). \tag{5.65}$$

Here $\times$ denotes the direct product of distributions and τ_a is translation by a in the x_1 direction.

Proof. Let $b_2, b_3, \cdots, b_n$ be fixed complex numbers such that $J(z_1, b_2, \cdots, b_n)$ does not vanish identically. By Proposition 5.3. the zeros of $J(z_1, b_2, \cdots, b_n)$ lie (except for a finite number) outside of an angular segment containing the real axis. Carleman's theorem (see [29]) this means that the density of zeros of $J(z_1, b_2, \cdots, b_n)$ is zero. Thus, $J(z_1, b_2, \cdots, b_n)$ is a polynomial in z_1 times a pure imaginary exponential in z_1, that is, for some a

$$J(z_1, b_2, \cdots, b_n) = \sum_{j=1}^{r} J_j(b_2, \cdots, b_n) z_1{}^j \exp(iaz_1), \tag{5.66}$$

where $J_j(b_2, \cdots, b_n)$ are certain complex numbers.

A priori, both a and r might depend on $b_2, \cdots, b_n$. If we multiply $J(z_1, b_2, \cdots, b_n)$ by $J(-z_1, b_2, \cdots, b_n)$ we obtain a new function $J^1(z_1, b_2, \cdots, b_n)$ of the form

$$J^1(z_1, b_2, \cdots, b_n) = \sum_{j=1}^{2r} J_j{}^1(b_2, \cdots, b_n) z_1{}^j. \tag{5.67}$$

Here again, r may depend on $b_2, \cdots, b_n$. But, if we apply Baire's category theorem, we can find an open set of $(b_2, \cdots, b_n)$ on which r is bounded, say by r_0. We expand both sides of (5.64) in power series in z_1 and it follows by analyticity that any $J_j{}^1$ must vanish if $j > 2r_0$. Thus, r is bounded.

Next, all the $J_j{}^1$ are entire functions of exponential type. It follows easily by comparing coefficients that the J_j are also entire functions in $\boldsymbol{E}'$. Hence, since a is bounded by the exponential type of J, a is an entire function of $(b_2, \cdots, b_n)$ and so must be a constant.

Using the result that $J^1(z_1, b_2, \cdots, b_n)$ is a polynomial of degree $\leqq 2r_0$ in z_1 we could conclude that a is a constant by applying the theorem of addition of supports (see [24]). It then again follows immediately that the J_j are entire functions of exponential type which lie in $\boldsymbol{E}'$. This completes the proof of Theorem 5.16.

Next, we want to find those differential difference operators which are C^∞ elliptic in x_1. We shall see that they are essentially differential operators in x_1. More generally, we have

THEOREM 5.17. *Let S be a differential difference operator in x_1 which is C^∞ elliptic in x_1. Then, in x_1, S is the composition of a differential operator with a translation.*

Proof. As in the proof of Theorem 5.16, we fix complex numbers $b_2, \cdots, b_n$ such that $J(z_1, b_2, \cdots, b_n)$ is identically zero. We could conclude the proof the same way as in the above theorem if we could prove that $J(z_1, b_2, \cdots, b_n)$ is a polynomial in z_1 times a pure imaginary exponential in z_1. This is a consequence of Theorem 5.8 and

LEMMA 5.18. *Let Q be an exponential polynomial in one variable with pure imaginary exponentials such that, if ${}_jz$ denotes its zeros, then*

$$\liminf |\mathfrak{I}({}_jz)|/\log(1 + |{}_jz|) = \infty.$$

Then Q is a polynomial times a pure imaginary exponential.

Proof. We write $Q(z) = \sum_{k=1}^{s} P_k(z)\exp(ib_kz)$, where P_k are polynomials not identically zero and b_k are real numbers with $b_1 < b_2 < \cdots < b_s$. If Q has a finite number of zeros the result is easy. If not, there exists a sequence of complex numbers c_l, with $|c_l| \to \infty$, which are zeros of Q. We show that this is impossible.

We may suppose for simplicity that $\mathfrak{I}c_l > 0$ for all l. Then we have

$$\begin{aligned} |P_s(c_l)| &= |\sum_{k<s} P_k(c_l)\exp(ic_l(b_k - b_s))| \\ &\leqq c(1 + |c_l|^m)\exp(-(\mathfrak{I}c_l)(b_{s-1} - b_s)) \end{aligned} \tag{5.68}$$

for suitable c, m which are independent of l. By (5.68) it follows that $P_s(c_l) \to 0$ which is impossible since P_s is a polynomial not identically zero. This completes the proof of Lemma 5.18 and hence of Theorem 5.17.

Remark. The result corresponding to Theorem 5.17 for weak C^∞ ellipticity does not hold as the example $(n = 1) S = d/dx - \tau$ shows. For then $J = iz - \exp(iz)$. For $J(z) = 0$ we have $(z = \xi + i\eta)$

$$|iz| = \exp(-\eta)$$

so

$$\xi^2 + \eta^2 = \exp(-2\eta). \tag{5.69}$$

Hence, if $\eta \leqq 0$, we have

$$-2\eta \geqq 2 \log |\xi|$$

so

$$|\eta| \geqq \log |\xi|.$$

If $\eta > 0$, then inequality (5.69) defines a compact set in the z plane so there are only a finite number of zeros of J there. Thus, S is weakly C^∞ elliptic. The fact that S is weakly C^∞ elliptic can also be seen easily directly. On the other hand, $S * \sum \delta_j^{(j)} = 0$ where $\delta_j^{(j)}$ is the j-th derivative of the unit mass at the point 1; this shows again that S is not C^∞ elliptic.

For $n > 1$, a similar computation shows that if

$$J = \exp(i(z_1 + z_2 + \cdots + z_n)) - (z_1^2 + z_2^2 + \cdots + z_n^2)$$

then S, which is a differential operator, is weakly C^∞ elliptic in all variables.

6. Unsolved problems and general remarks. In addition to the problems and remarks stated in the text, we have the following:

1. One of the most important problems is to give, in case $n = 1$, conditions on the zeros of a $J \in \boldsymbol{E}'$ to insure that J should be slowly decreasing. Certain sufficient conditions are known if the zeros are "close to" the integers (see, e.g., [22]). But all these results are obtained by reducing the question to the known case of $\sin z$. Certainly a necessary and sufficient condition would be of great interest. In this connection we have one partial result:

PROPOSITION 6.1. *Suppose J has a sequence of zeros a_j of multiplicities r_j which satisfy*

$$\liminf r_j/(|\mathfrak{I} a_j| + \log |\mathfrak{R} a_j|) = \infty. \tag{6.1}$$

Then J is not slowly decreasing.

Proof. We may suppose as usual that $|J(z)| \leqq 1$ for $z \in R$; suppose for simplicity that J is of exponential type $\leqq 1$. Then the Phragmén-Lindelöf theorem (see [29] tells us that on the line $\mathfrak{I} z = \mathfrak{I} a_j$ we have $|J(z)| \leqq \exp(|\mathfrak{I} a_j|)$. We now apply Bernstein's theorem (see, e.g., [14]) which shows that

$$|J^{(k)}(a_j)| \leqq \exp(|\mathfrak{I} a_j|). \tag{6.2}$$

Now, let us examine the Taylor expansion of J about a_j:

$$
\begin{aligned}
|J(z)| &= |\sum J^{(k)}(a_j)(z-a_j)^k/k!| \\
&\leqq \exp(|\mathfrak{I}a_j|) \sum_{k \geqq r_j} |z-a_j|^k/k! \\
&= \exp(|\mathfrak{I}a_j|)\,|z-a_j|^{r_j} \sum_{k \geqq r_j} |z-a_j|^{k-r_j}/k! \qquad (6.3)\\
&\leqq (r_j!)^{-1} \exp(|\mathfrak{I}a_j|)\,|z-a_j|^{r_j} \sum_{k \geqq 0} |z-a_j|^k/k! \\
&\leqq (r_j!)^{-1} \exp(|\mathfrak{I}a_j|)\,|z-a_j|^{r_j} \exp|z-a_j|.
\end{aligned}
$$

Thus, $|J(z)| \leqq |\mathfrak{R}a_j|^{-l}$ whenever

$$|z-a_j|^{r_j} \leqq |\mathfrak{R}a_j|^{-l} \exp(-|\mathfrak{I}a_j|) - |z-a_j|)(r_j)!,$$

that is, whenever

$$r_j \log|z-a_j| \leqq -l \log|\mathfrak{R}a_j| - |\mathfrak{I}a_j| - |z-a_j| + \log r_j!.$$

Using Stirling's formula, this is true whenever

$$r_j \log|z-a_j| + |z-a_j| - r_j \log r_j \leqq -l \log|\mathfrak{R}a_j| - |\mathfrak{I}a_j| - \text{const.}\, r_j.$$

We now use our hypothesis (6.1) and we obtain the result that $|J(z)| \leqq |\mathfrak{R}a_j|^{-l}$ whenever j is large and

$$r_j \log|z-a_j| + |z-a_j| - r_j \log r_j \leqq -\text{const.}\, r_j. \qquad (6.4)$$

It is clear that (6.4) is satisfied whenever $|z-a_j| \leqq br_j$ for a suitable $b > 0$. This shows that J cannot be slowly decreasing, which is our assertion.

It is clear that inequality (6.1) can be slightly ameliorated, and moreover, that we don't need r_j zeros at a_j, but only sufficiently close to a_j (in that case the first r_j Taylor coefficients of J at a_j are very small and the others are handled as before). However, even an "almost sufficient" condition of this type for J to be slowly decreasing seems very difficult.

2. Presumably, we could use the results of Schwartz [25] to show that if J has real distinct zeros, is slowly decreasing, and if the index of condensation of the zeros is zero, then the quotient space $\boldsymbol{E}'/J\boldsymbol{E}'$ could be described as the space of slowly increasing functions (i.e. sequences) on the zeros of J. This method could also be slightly modified in case the zeros of J are not real and distinct. The problem arises as to what is the quotient space $\boldsymbol{E}'/J\boldsymbol{E}'$ in general. This problem is connected with my fundamental principle (see [15], [16]) and has been solved for some slowly decreasing J even in case $n > 1$. But I know of no answer to the question in case J is not slowly decreasing even if $n = 1$.

3. In connection with the results of Section 2, can Theorems 2.8, 2.9 be completed to describe completely $S * \mathfrak{D}$ and to decide when $S * \mathfrak{D}' \supset T * \mathfrak{D}'$? In particular, is $S * \mathfrak{D}$ bornologic?

4. Is $\mathfrak{D} * \mathfrak{D} = \mathfrak{D}$? Is $\mathfrak{D} * \mathcal{E} = \mathcal{E}$, or even $\mathfrak{D} * \mathfrak{D}' = \mathcal{E}$?

5. In Section 4 we proved that $\bigcap_{S \in \mathcal{E}'} S * \mathfrak{D}' = \bigcap_{S \in \mathcal{E}'} S * \mathcal{E} = A$. We can improve this result slightly as follows: Let M be any monotonic increasing sequence for which A_M is not quasi analytic. Define $\mathcal{E}_M$ as the subspace of $f \in \mathcal{E}$ which, together with all their derivatives, satisfy inequality (4.1) on every compact set. We could use the methods of Section 4 to show that $\bigcap_{S \in \mathcal{E}'} S * \mathcal{E}_M = A$. But, is $\bigcap_{S \in \mathcal{E}'} S * A = A$?

6. Closely related to problem 5 is the following problem: Let $\mathfrak{D}_M = \mathfrak{D} \cap \mathcal{E}_M$. We introduce a natural topology in $\mathfrak{D}_M$ as in [14]; call $\mathfrak{D}_M$ the dual of $\mathfrak{D}_M$. Given $S \in \mathcal{E}'$, can we always find an M such that the equation $S * T = \delta$ has a solution $T \in \mathfrak{D}'_M$, or even, can we find an M such that $S * \mathfrak{D}'_M \supset \mathfrak{D}'$?

We give now an example to show that this is not the case. Let $n = 1$ and define the Fourier transform J of S to be

$$J(z) = \prod \cos(z/j!\, j \log^2 j)^{j!}. \tag{6.5}$$

THEOREM 6.2. *S is not invertible for any $\mathfrak{D}'_M$ for which $\mathcal{E}_M$ is Carleman non quasi-analytic. In fact, for no such M does there exist a $T \in \mathfrak{D}'_M$ which satisfies $S * T = \delta$.*

Proof. If there were such a T, then for B any set in $\mathfrak{D}$ such that $S * B$ is bounded in $\mathcal{E}'$, we must have B bounded in $\mathcal{E}'_M$. For,

$$B = (S * T) * B = T * (S * B)$$

and so is bounded in $\mathcal{E}'_M$ by the continuity of convolution.

Let us set $n_j = j!\, j \log^2 j$. We return to the notation of the proof of Theorem 2.1. We set

$$L_j(z) = \exp(n_j/j \log j) H_{[n_j/j \log j]}(z - n_j) \tag{6.6}$$

and call $B = \{L_j\}$. Then I claim that JB is bounded in $\boldsymbol{E}'$ but B is not bounded in $\boldsymbol{E}'_M$ for any M such that $\mathcal{E}_M$ is non quasi-analytic.

The fact that JB is bounded in $\boldsymbol{E}'$ is seen as follows: JL_j is certainly small far away from n_j, for $|J(x)| \leqq 1$ for $x \in R$ and $|L_j(x)| \leqq R$ when $|x - n_j| \geqq n_j/j \log j$.

Now, $\cos(z/n_j)$ vanishes when $z = n_j$. Moreover, it behaves linearly near n_j with slope $1/n_j$. Thus we ask when is $(x/n_j)^{j!} = \exp(-n_j/j\log j)$, that is when is

$$j!(\log x - \log n_j) = -n_j/j\log j$$

$$j!(\log x - \log j - 2\log\log j - \log j!) = -j!\log j$$

$$\begin{aligned}\log x &= \log j + 2\log\log j + \log j! - \log j\\ x &= j!\log^2 j\\ &= j!\,j\log^2 j/j\\ &= n_j/j.\end{aligned}$$

It follows easily that JB is bounded in $\boldsymbol{E}'$.

Next we show that B is not bounded in $\boldsymbol{E}'_M$ for $\mathcal{E}_M$ non quasi-analytic. Now, the bounded sets of $\boldsymbol{E}'_M$ can be described as follows: All the functions are of fixed exponential type and are majorized on R by a continuous monotonic increasing function $H \geqq 1$ for which

$$\int [\log H(x)/(1+x^2)]\,dx < \infty.$$

In particular, if B were bounded,

$$\log H(n_j) \geqq n_j/j\log j.$$

But the n_j are lacunary enough so that

$$\begin{aligned}\int_{n_j}^{n_{j+1}} [\log H(n_j)/(1+x^2)]\,dx &\geqq \log H(n_j)/2n_j\\ &\geqq 1/2j\log j.\end{aligned}$$

Thus,

$$\int [\log H(x)/(1+x^2)]\,dx \geqq \tfrac{1}{2}\sum 1/j\log j = \infty.$$

This contradiction completes the proof of Theorem 6.2.

7. In Section 5 we used lacunary series of exponentials to construct examples. It should be of interest to study these series in more detail in case $n > 1$.

8. The results of Section 2 are non constructive. In fact, it would be of interest to give a constructive method for finding an elementary solution for S in case S is invertible. It is not difficult to give such a procedure in

case S is a partial differential difference operator. Results of this kind are of importance in studying equations depending on a parameter (see, e. g., [28]).

9. Finally, the problem arises as to extend the results of this paper to systems of convolution equations. In case the determinant of the system $\not\equiv 0$, then we can use the methods of this paper together with Cramer's rule to obtain the corresponding results. But, in case the determinant is $\equiv 0$, the problem seems to be extremely difficult. The simplest example is probably the following: Let S_1 and S_2 be slowly decreasing; when can we solve $S_1 * T = W_1$, $S_2 * T = W_2$, where $W_1 W_2 \in \mathfrak{D}'$ are given? Clearly a necessary condition is $S_1 * W_2 = S_2 * W_1$. But, even for differential operators this condition cannot be sufficient, for J_1 and J_2 may not be relatively prime. If we assume J_1, J_2 relatively prime, then is $S_1 * W_2 = S_2 * W_1$ sufficient? Since S_1 is invertible, we can reduce the problem to the case $W_1 = 0$. The problem becomes: If $S_1 * W_2 = 0$, can we find a T such that $S_2 * T = W_2$ and $S_1 * T = 0$? This suggests that we try to apply the methods of this paper to the subspace of $\mathfrak{D}'$ which is the kernel of S_1. For this, we need a "good" description of the Fourier transform of the dual of this kernel which is $\mathfrak{D}/S_1 * \mathfrak{D}$. This is just Problem 2.

In this connection we should add that the questions of simultaneous C^∞ (entire) ellipticity in all variables for a system of differential operators can be reduced to the case of a single differential operator as has been shown by the work of L. Hörmander ("Differentiability properties of solutions of system of differential equations," *Arkiv för Matematik*, vol. 3 (1958), pp. 527-535, and C. Lech, "A metric property of the zeros of a complex polynomial ideau," *Arkiv för Matematik*, vol. 3 (1958), pp. 543-554.)

However, the method of Hörmander and Lech fails in the case of systems of convolution equations. For, we can easily construct $(n = 1)$ $J_1, J_2 \in \boldsymbol{E}'$ whose common zeros are infinite in number and lie outside of some angle containing R. Thus, by a result of Schwartz (see [27]) every $T \in \mathfrak{D}'$ which satisfies $S_1 * T = 0$, $S_2 * T = 0$ must be an entire function. But, by Carleman's theorem (see [29]) there is no J in $\boldsymbol{E}'$ which has infinitely many zeros all of which lie outside of some angle containing R. Thus, we cannot reduce the above problem for ideals to the problem for a single $S \in \mathcal{E}'$.

Moreover, I do not know if the methods of Hörmander and Lech can be extended to the problem of ellipticity in x_1.

Appendix, added in proof. We indicate some of the progress made since this paper was written: (The numbers correspond to the numbers used in Section 6.)

1. We can produce an example of a J which is slowly decreasing $(n=1)$ but such that the orders of the zeros of J are unbounded.

2. In case J is slowly decreasing we can give $(n=1)$ necessary and sufficient conditions in order that E'/JE' should be isomorphic with the space of slowly increasing sequences on the zeros of J. In case J has multiple zeros, a similar result is possible.

4. In case $n > 1$ I can produce an example to show that $\mathfrak{D} * \mathfrak{D} \neq \mathfrak{D}$.

7. The study of these lacunary series leads to an extension of the Fabry gap theorem to $n > 1$.

9. The results on partial ellipticity can be extended completely to systems of partial differential equations and to a very few other convolution systems (see [15], [16]).

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HOMOMORPHISMS OF COMMUTATIVE BANACH ALGEBRAS.*[1]

By W. G. Bade and P. C. Curtis, Jr.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be commutative Banach algebras, and ν be an arbitrary (not necessarily continuous) homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. This paper is a study of continuity properties of ν which arise from the algebraic structure of $\mathfrak{A}$. The main results are grouped around the following topics:

(a) The degree of discontinuity which ν may have on the set of idempotents in $\mathfrak{A}$.

(b) The localization of the discontinuity of ν to a finite set of points of the structure space $\Phi_{\mathfrak{A}}$ of $\mathfrak{A}$ when $\mathfrak{A}$ is a regular algebra in the sense of Silov.

(c) The question of the existence of discontinuous homomorphisms (or equivalently, the existence of incomplete multiplicative norms) on the algebra $C(\Omega)$ of all continuous functions on a compact Hausdorff space.

(d) The construction of algebras which are not normed algebras under any norm.

If $\nu\colon \mathfrak{A} \to \mathfrak{B}$, then the function $|x| = \|\nu(x)\|$, $x \in \mathfrak{A}$, is a multiplicative semi-norm on $\mathfrak{A}$. Conversely (cf. Section 1), every multiplicative semi-norm is the norm of a homomorphism. Thus all our results on continuity of homomorphisms could be stated equivalently in terms of continuity properties of multiplicative semi-norms on $\mathfrak{A}$. We have chosen the homomorphism approach as it reveals the methods more clearly.

Section 1 contains preliminary material concerning adjunction of units and the relation of multiplicative semi-norms to homomorphisms. The first theorem of Section 2 is the key result of the paper: If a bounded sequence $\{g_n\}$ of elements of $\mathfrak{A}$ is separated by orthogonal relative units (elements h_n of $\mathfrak{A}$ satisfying $g_n h_n = g_n$, $h_n h_m = 0$, $m \neq n$), then under any homomorphism

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$\nu\colon \mathfrak{A} \to \mathfrak{B}$, the norms of the elements $\nu(g_n)$ in $\mathfrak{B}$ cannot grow faster than the norms of the relative units h_n in $\mathfrak{A}$. This result is best possible. From this theorem we obtain an important property of ν on the set $\mathfrak{P}$ of idempotents of $\mathfrak{A}$: there exists a constant M such that

$$\| \nu(p) \| \leqq M \| p \|^2, \; p \in \mathfrak{P}.$$

Again the result is sharp. Thus if $\mathfrak{P}$ is a bounded set in $\mathfrak{A}$, it remains bounded under any homomorphism.

In Section 3 the algebra $\mathfrak{A}$ is supposed to be semi-simple, with unit, and to be regular in the sense of Silov. Regarding $\mathfrak{A}$ as an algebra of continuous functions on its structure space $\Phi_{\mathfrak{A}}$, it is shown that if $\nu\colon \mathfrak{A} \to \mathfrak{B}$, then there is a finite set F of points of $\Phi_{\mathfrak{A}}$ such that ν is continuous on the ideal of functions in $\mathfrak{A}$ which vanish identically in any given neighborhood of F. Examples are given where the set F is not empty.

The results of Section 3 are considerably strengthened in Section 4 for the case of the special Banach algebra $C(\Omega)$. Any homomorphism $\nu\colon C(\Omega) \to \mathfrak{B}$ has a decomposition $\nu = \mu + \lambda$, where μ is a continuous homomorphism of $C(\Omega)$, coinciding with ν on a dense subalgebra, and λ maps into the radical of $\mathfrak{B}$. Moreover,

$$\overline{\nu(C(\Omega))} = \mu(C(\Omega)) \oplus \lambda\overline{(C(\Omega))},$$

the direct sum being topological. If the radical of $\mathfrak{B}$ is nil, then $\lambda \equiv 0$ and ν is therefore continuous. It is shown that the existence of an incomplete multiplicative norm on $C(\Omega)$, or of a discontinuous homomorphism, is equivalent to the existence of a non-trivial homomorphism of some maximal ideal of $C(\Omega)$ into a radical Banach algebra. We know of no example of such a homomorphism.

Section 5 deals with the non normability of certain quotient algebras. Explicitly, if $\mathfrak{A}$ is a semi-simple regular algebra with unit, $\varphi_0 \in \Phi_{\mathfrak{A}}$, and $\mathfrak{J}(\varphi_0)$ is the ideal of all functions in $\mathfrak{A}$ which vanish in a neighborhood of φ_0, then the algebra $\mathfrak{A}/\mathfrak{J}(\varphi_0)$ is not normable whenever φ_0 is the limit of a sequence in $\Phi_{\mathfrak{A}}$. Section 6 contains a discussion of a Banach algebra due to C. Feldman [2] which shows that the theorems of Section 1 cannot be improved. It also provides an example of an algebra with one-dimensional radical which admits two inequivalent complete multiplicative norms. This shows that the theorem of Gelfand [3] to the effect that semi-simple commutative Banach algebras have unique Banach algebra topologies cannot be generalized even to algebras with finite dimensional radical.

Our work is related in spirit to the important results of Gelfand [3],

Rickart [8], Silov [9, Theorem 8], Yood [11], and others on continuity of homomorphisms and uniqueness of norms for Banach algebras. However, the methods and results are very different. Typical results of their work are the theorem of Rickart that any homomorphism of a Banach algebra into a commutative semi-simple Banach algebra is continuous. In these results essential use is made of completeness and semi-simplicitly of the range of the homomorphism. In our work the special assumptions are placed on the domain algebra, the range being arbitrary. The conjecture[2] that on $C(\Omega)$ every multiplicative norm $\|\cdot\|$ is complete and equivalent to the supremum norm arises naturally from a theorem of Kaplansky [5], that one necessarily has $\|x\| \geqq \sup_\Omega |x(\omega)|$, $x \in C(\Omega)$.

1. Preliminaries. Let $\mathfrak{A}$ be a commutative Banach algebra and ν be a homomorphism of $\mathfrak{A}$ into a Banach algebra $\mathfrak{B}$. Before beginning the study of general properties of ν, it is convenient for technical reasons to make some simplifications. First, by confining attention to $\overline{\nu(\mathfrak{A})}$ we may always assume that ν maps $\mathfrak{A}$ into a commutative Banach algebra $\mathfrak{B}$. Similarly, if $\mathfrak{A}$ has a unit e, we may assume that $\mathfrak{B}$ has a unit e' and that $\nu(e) = e'$. If $\mathfrak{A}$ does not have a unit, then we may form $\mathfrak{A}' = \mathfrak{A} \oplus \{\lambda e\}$ in the usual way. If $\mathfrak{B}$ has a unit, let $\mathfrak{B}' = \mathfrak{B}$; otherwise let $\mathfrak{B}' = \mathfrak{B} \oplus \{\lambda e'\}$. For $x' \in \mathfrak{A}'$, $\|x'\| = \|x\| + |\lambda|$, where $x' = x + \lambda e$, $x \in \mathfrak{A}$. If e' is adjoined to $\mathfrak{B}$ to form $\mathfrak{B}'$, then $\mathfrak{B}'$ is normed in the same way. In either case it is clear that any homomorphism ν of $\mathfrak{A}$ into $\mathfrak{B}$ may be extended to a homomorphism ν' of $\mathfrak{A}'$ into $\mathfrak{B}'$ and that continuity properties of ν are uneffected by this extension. In light of these remarks we shall always assume that $\mathfrak{A}$ and $\mathfrak{B}$ are commutative Banach algebras with units e, e' respectively and that $\nu(e) = e'$.

The study of homomorphisms ν of a Banach algebra $\mathfrak{A}$, commutative or not, is equivalent to the study of multiplicative semi-norms as the following makes clear.

1.1. *Definition.* Let $\mathfrak{A}$ be a Banach algebra with unit. A *multiplicative semi-norm* on $\mathfrak{A}$ is a function $|\cdot|$ on $\mathfrak{A}$ to $[0,\infty)$ satisfying

(i) $|x+y| \leqq |x| + |y|$, $x, y \in \mathfrak{A}$,

(ii) $|xy| \leqq |x|\,|y|$, $x, y \in \mathfrak{A}$,

(iii) $|\alpha x| = |\alpha|\,|x|$, $x \in \mathfrak{A}$, α scalar,

(iv) $|e| = 1$.

[2] **An early paper of Gelfand and Naimark contains a statement [4, Lemma 2] which, with Kaplansky's theorem, would imply this conjecture. However, the proof contains a serious gap.**

If in addition $|x| = 0$ implies $x = 0$, then $|\cdot|$ will be called a *multiplicative norm.*

Clearly, for a multiplicative semi-norm,

$$||x| - |y|| \leqq |x - y|, \text{ hence } |x| = |y| \text{ if } |x - y| = 0.$$

1.2. THEOREM. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Branch algebras with units e, e' respectively. If ν is a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ with $\nu(e) = e'$, then the function $|x| = \|\nu(x)\|$, $x \in \mathfrak{A}$, is a multiplicative semi-norm on $\mathfrak{A}$. Conversely, if $|\cdot|$ is a multiplicative semi-norm on $\mathfrak{A}$, then there exists ν of $\mathfrak{A}$ into a Banach algebra $\mathfrak{B}$ such that $|x| = \|\nu(x)\|$, $x \in \mathfrak{A}$.*

Proof. The first assertion is clear. To prove the second we note that the set $\mathfrak{R} = \{x \mid |x| = 0\}$ is an ideal and $|\cdot|$ is constant on the cosets $[x + \mathfrak{R}]$ of $\mathfrak{A}/\mathfrak{R}$. Thus $\mathfrak{A}/\mathfrak{R}$ is a normed algebra under the norm $|x + \mathfrak{R}| = |x|$. Let ν be the natural homomorphism of $\mathfrak{A}$ into the completion of $\mathfrak{A}/\mathfrak{R}$ in this norm. It has the required properties.

2. The main boundedness theorem. The first theorem of this section is the main technical device of the paper. We are indebted to Y. Katznelson for suggestions which greatly shortened the original proof.

2.1. THEOREM. *Let $\mathfrak{A}$ be a commutative Banach algebra and ν be a homomorphism of $\mathfrak{A}$ into a Banach algebra $\mathfrak{B}$. If $\{g_n\}$ and $\{h_n\}$ are sequences from $\mathfrak{A}$ satisfying*

(i) $g_n h_n = g_n$, $n = 1, 2, \cdots$,

and

(ii) $h_m h_n = 0$, $m \neq n$,

then

$$\sup \|\nu(g_n)\| / \|g_n\| \, \|h_n\| < \infty.$$

Proof. Suppose on the contrary that

$$\limsup \|\nu(g_n)\| / \|g_n\| \, \|h_n\| = +\infty.$$

We may suppose $\|g_n\| = 1$, $n = 1, 2, \cdots$. Clearly $\|h_n\| \geqq 1$ by (i). It will be shown that a suitable linear combination of the elements h_n must map into an element of infinite norm. Select distinct elements q_{ij}, $i, j = 1, 2, \cdots$, from the sequence $\{g_n\}$ such that

$$(*) \qquad \|\nu(q_{ij})\| \geqq 4^{i+j} \|p_{ij}\|, \qquad i, j = 1, 2, \cdots,$$

where p_{ij} is the relative unit h_m corresponding to $g_m = q_{ij}$. Define

$$f_i = \sum_{j=1}^{\infty} q_{ij}/2^j, \qquad i = 1, 2, \cdots.$$

The equation $p_{ij}f_i = 2^{-j}q_{ij}$ and (*) show $\nu(f_i) \neq 0$. For each integer i select an integer j_i so large that $2^{j_i} > \| \nu(f_i) \|$ and define

$$y = \sum_{i=1}^{\infty} p_{ij_i}/2^i \| p_{ij_i} \|.$$

It follows from (i) that

$$f_i y = q_{ij_i}/2^{(i+j_i)} \| p_{ij_i} \|, \qquad i = 1, 2, \cdots.$$

Thus, using (*),

$$\| \nu(y) \| \, \| \nu(f_i) \| \geqq \| \nu(f_i y) \| \geqq 2^{(i+j_i)} > 2^i \| \nu(f_i) \|.$$

Thus $\| \nu(y) \| > 2^i$ for every integer i.

2.2. Corollary. *Let ν be an arbitrary homomorphism of the commutative Banach algebra $\mathfrak{A}$ into a Banach algebra $\mathfrak{B}$. If $\{p_n\}$ is a sequence of orthogonal idempotents in $\mathfrak{A}$, i.e. $p_m p_n = 0$ for $m \neq n$, then there exists a constant M such that*

$$\| \nu(p_n) \| \leqq M \| p_n \|^2, \qquad n = 1, 2, \cdots.$$

Proof. The result follows by taking $g_n = h_n = p_n$ in Theorem 2.1.

The next theorem shows that the constant M may be chosen independent of the sequence $\{p_n\}$.

2.3. Theorem. *Let ν be a homomorphism of the commutative Banach algebra $\mathfrak{A}$ into a Banach algebra $\mathfrak{B}$ and let $\mathfrak{P}$ denote the set of idempotents in $\mathfrak{A}$. There exists a constant M such that*

$$\| \nu(p) \| \leqq M \| p \|^2, \qquad p \in \mathfrak{P}.$$

If $\mathfrak{P}$ is a bounded set in $\mathfrak{A}$, its image under any homomorphism is bounded.

Proof. By the remarks of Section 1 we may suppose $\mathfrak{A}$ has a unit e and that $\nu(e)$ is the unit of $\mathfrak{B}$. Supposing the theorem false, we shall construct an orthogonal sequence which contradicts Corollary 2.2.

Let $\mathfrak{P}_1$ denote the set of $p \in \mathfrak{P}$ such that

$$\sup_{q \leqq p} \| \nu(q) \| / \| q \|^2 = +\infty.$$

By assumption $e \in \mathfrak{P}_1$. Note that if $p \in \mathfrak{P}_1$ and $q \leqq p$, then either q or $p - q$ is in $\mathfrak{P}_1$. For otherwise, there is a constant K such that $\| \nu(r) \| \leqq K \| r \|^2$

for all $r \leqq q$ and $r \leqq p-q$. If $s \leqq p$, we may write $s = sq + s(p-q)$, so

$$\| \nu(s) \| \leqq K[\| sq \|^2 + \| s(p-q) \|^2] \leqq K \| s \|^2 [\| q \|^2 + \| p-q \|^2],$$

contradicting the assumption that p belongs to $\mathfrak{P}_1$.

For purposes of an induction let r_1 belong to $\mathfrak{P}_1$ and choose $q_1 \leqq r_1$ such that

$$\| \nu(q_1) \| / \| q_1 \|^2 > 16 \| r_1 \|^4 [2 + 2 \| \nu(r_1) \| / \| r_1 \|^2].$$

Then

$$\begin{aligned} \| \nu(r_1 - q_1) \| / \| r_1 - q_1 \|^2 &> [\| \nu(q_1) \| - \| \nu(r_1) \|] / [\| r_1 \| \, \| q_1 \| + \| q_1 \|]^2 \\ &\geqq [\| \nu(q_1) \| / 4 \| r_1 \|^2 \| q_1 \|^2] - [\| \nu(r_1) \| / \| r_1 \|^2] \\ &> 4 \| r_1 \|^2 [2 + \| \nu(r_1) \| / \| r_1 \|^2]. \end{aligned}$$

Let r_2 be the member of the pair q_1, $r_1 - q_1$, which is in $\mathfrak{P}_1$. Then clearly,

$$\| \nu(r_2) \| / \| r_2 \|^2 > 4 \| r_1 \|^2 [2 + \| \nu(r_1) \| / \| r_1 \|^2].$$

By an exactly similar arguments we obtain inductively a sequence $\{r_k\}$ of idempotents in $\mathfrak{P}_1$ such that $r_{k+1} \leqq r_k$ and

$$\| \nu(r_k) \| / \| r_k \|^2 > 4 \| r_{k-1} \|^2 [k + \| \nu(r_{k-1}) \| / \| r_{k-1} \|^2], \qquad k = 2, 3, \cdots.$$

Define $p_k = r_k - r_{k+1}$. Then $p_k p_l = 0$ if $k \neq l$ and

$$\begin{aligned} \| \nu(p_k) \| / \| p_k \|^2 &\geqq [\| \nu(r_{k+1}) \| / 4 \| r_k \|^2 \| r_{k+1} \|^2] - [\| \nu(r_k) \| / \| r_k \|^2] \\ &> k + 1, \qquad\qquad k = 2, 3, \cdots, \end{aligned}$$

contradicting Corollary 2.2.

3. Homomorphisms of regular algebras. In this section $\mathfrak{A}$ will be a commutative semi-simple Banach algebra with unit which is regular in the sense of Silov [9]. We shall regard $\mathfrak{A}$ via the Gelfand isomorphism as an algebra of continuous functions on its structure sapce $\Phi_{\mathfrak{A}}$.[3] Recall that the property of being regular is equivalent to the condition that given any two disjoint closed sets F_1 and F_2 in $\Phi_{\mathfrak{A}}$, there exists a function in $\mathfrak{A}$ which is zero on F_1 and one on F_2 (cf. [9] or Loomis [7, p. 84]). We shall show that if ν is any homomorphism of $\mathfrak{A}$ into a Banach algebra, there exists a finite set F of points of $\Phi_{\mathfrak{A}}$ such that for any neighborhood V of F the restriction of ν to the ideal $\mathfrak{J}(V) = \{f \in \mathfrak{A} \mid f(V) = 0\}$ is continuous. We shall also obtain information as to how the norm of ν on $\mathfrak{J}(V)$ depends on the neighborhood V.

[3] The topology of $\mathfrak{A}$ is always the Banach algebra topology of $\mathfrak{A}$, rather than the relative sup norm topology, except, of course, when they coincide.

3.1. *Definition.* We denote by $\mathfrak{G}$ the family of all open sets $E \subseteq \Phi_{\mathfrak{A}}$ with the property that

$$\sup \| \nu(g) \| / \| g \| \, \| h \| = M_E < \infty$$

for all functions g and h having carriers in E and such that $gh = g$.

We shall show that $\mathfrak{G}$ contains a maximal open set whose complement is finite. This will be accomplished through a sequence of lemmas. The first of these, a direct consequence of the main boundedness theorem of Section 2, shows the existence of many open sets in $\mathfrak{G}$.

3.2. LEMMA. *If $\{E_n\}$ is any sequence of disjoint open sets in $\Phi_{\mathfrak{A}}$, then $E_n \in \mathfrak{G}$ for all sufficiently large n.*

Proof. If the lemma is false there exists an infinite sequence $\{E_m\}$ of disjoint open sets and functions g_m, h_m in $\mathfrak{A}$, whose carriers lie in E_m such that

(i) $\| g_m \| = 1$,

(ii) $g_m h_m = g_m$,

and

(iii) $\| \nu(g_m) \| > m \| h_m \|$,

which contradicts Theorem 2.1.

The next task is to prove that $\mathfrak{G}$ is closed under arbitrary unions. Several lemmas will be required.

3.3. LEMMA. *Let E_1 and E_2 belong to $\mathfrak{G}$. If G is an open set such that $\bar{G} \subseteq E_2$, then $E_1 \cup G$ is in $\mathfrak{G}$.*

Proof. By regularity we can choose a function $u_1 \in \mathfrak{A}$ which is one on a neighborhood of E_2' and zero on a neighborhood of $\bar{G}$. Let $u_2 = 1 - u_1$. Since $\operatorname{car}(u_1) \cap \bar{G} = \phi$ and $\operatorname{car}(u_2) \cap E_2' = \phi$ we can find functions v_1 and v_2 in $\mathfrak{A}$ such that

$$u_1 v_1 = u_1, \qquad \operatorname{car}(v_1) \cap \bar{G} = \phi,$$

$$u_2 v_2 = u_2, \qquad \operatorname{car}(v_2) \cap E_2' = \phi.$$

Let $H = E_1 \cup G$ and suppose $\operatorname{car}(g) \subseteq H$, $\operatorname{car}(h) \subseteq H$ and $gh = g$. Then

$$\operatorname{car}(gu_i) \subseteq E_i, \qquad \operatorname{car}(hv_i) \subseteq E_i, \qquad i = 1, 2,$$

and

$$gu_i = ghu_iv_i = (gu_i)(hv_i), \qquad i = 1, 2.$$

Since E_1 and E_2 belong to $\mathfrak{G}$,

$$\begin{aligned}\|\nu(g)\| &\leqq \|\nu(gu_1)\| + \|\nu(gu_2)\| \\ &\leqq M_{E_1}\|gu_1\|\,\|hv_1\| + M_{E_2}\|gu_2\|\,\|hv_2\| \\ &\leqq \{M_{E_1}\|u_1\|\,\|v_1\| + M_{E_2}\|u_2\|\,\|v_2\|\}\|g\|\,\|h\|,\end{aligned}$$

showing H belongs to $\mathfrak{G}$.

3.4. COROLLARY. *If $E_1, E_2 \in \mathfrak{G}$ and G is open with $\bar{G} \subseteq E_1 \cup E_2$, then $G \in \mathfrak{G}$.*

Proof. The closed set $F = E_1' \cap \bar{G}$ is contained in E_2. Let U be open with $F \subseteq U \subseteq \bar{U} \subseteq E_2$. Then $G \subseteq E_1 \cup U$ which belongs to $\mathfrak{G}$ by Lemma 3.3. Thus $G \in \mathfrak{G}$.

3.5. LEMMA. *If $E_1, E_2 \in \mathfrak{G}$, then $E_1 \cup E_2 \in \mathfrak{G}$.*

Proof. Suppose $E_1 \cup E_2 \notin \mathfrak{G}$. We note that if F is closed and $F \subseteq E_1 \cup E_2$, then $G = (E_1 \cup E_2) \sim F$ is also not in $\mathfrak{G}$. For choose open sets U and V such that

$$F \subseteq V \subseteq \bar{V} \subseteq U \subseteq \bar{U} \subseteq E_1 \cup E_2.$$

Then $U \in \mathfrak{G}$ by Corollary 3.4. If $G \in \mathfrak{G}$, then by Lemma 3.3, $E_1 \cup E_2 = G \cup V \in \mathfrak{G}$, contrary to assumption.

Now if $E_1 \cup E_2 \notin \mathfrak{G}$, we can find g_1, h_1, such that $g_1 = g_1 h_1$, $\mathrm{car}(h_1) \subseteq E_1 \cup E_2$ and

$$\|\nu(g_1)\| > \|g_1\|\,\|h_1\|.$$

Pick U_1 open such that $\mathrm{car}\,(h_1) \subseteq U_1 \subseteq \bar{U}_1 \subseteq E_1 \cup E_2$. Then by the remark above $G_2 = (E_1 \cup E_2) \sim \bar{U}_1 \notin \mathfrak{G}$, and hence there exist $g_2, h_2, g_2 h_2 = g_2$, and $\mathrm{car}(h_2) \subseteq G_2$, satisfying

$$\|\nu(g_2)\| \geqq 2\,\|g_2\|\,\|h_2\|.$$

Continuing inductively we obtain sequences $\{g_n\}$, $\{h_n\}$ such that

(1) $g_n h_n = g_n$,

(2) $\|\nu(g_n)\| \geqq n\,\|g_n\|\,\|h_n\|$,

(3) the carriers of the h_n lie in disjoint open sets. This contradiction of Theorem 2.1 completes the proof.

3.6. COROLLARY. *$\mathfrak{G}$ is closed under arbitrary unions.*

Proof. Let $E_0 = \cup E_\alpha$, where $E_\alpha \in \mathfrak{G}$. Suppose $E_0 \notin \mathfrak{G}$. Then if F is closed and $F \subseteq E_0$, we note $E_0 \sim F \notin \mathfrak{G}$. For by compactness F is covered by a finite union E_1 of sets in $\mathfrak{G}$, i.e. a set in $\mathfrak{G}$. Thence $E_0 = (E_0 \sim F)$

$\cup E_1 \in \mathfrak{G}$. Now a repetition of the construction of the last proof yields a contradiction.

3.7. THEOREM. *Let $\mathfrak{A}$ be a commutative semi-simple regular Banach algebra with unit and let ν be an arbitrary homomorphism of $\mathfrak{A}$ into a Banach algebra. There exists a finite set F of points of $\Phi_{\mathfrak{A}}$ and a constant M such that*

$$\| \nu(g) \| \leqq M \| g \| \| h \|$$

for all functions g and h in $\mathfrak{A}$ having carriers in $\Phi_{\mathfrak{A}} \sim F$ and such that $gh = g$.

Proof. By Corollary 3.6 the class $\mathfrak{G}$ contains a maximal open set G_0. Let F be its complement. If F is infinite, we may separate a sequence of its elements by disjoint open sets E_n. By Lemma 3.2 $E_n \in \mathfrak{G}$ for large n. Thus G_0 must contain points of its complement. This contradiction shows F is finite.

3.8. *Definition.* The finite set F of Theorem 3.7 will be called the *singularity set* of ν.

If V is an open set in $\Phi_{\mathfrak{A}}$ we write $\mathfrak{J}(V) = \{f \in \mathfrak{A} \mid f(V) = 0\}$.

3.9. COROLLARY. *If V is any neighborhood of the singularity set F of ν, then the restriction of ν to $\mathfrak{J}(V)$ is continuous, and*

$$\| \nu(f) \| \leqq M \| f \| \| h \|, \qquad f \in \mathfrak{J}(V),$$

where h is any function in $\mathfrak{A}$ which is one on V' and which vanishes in a neighborhood of F.

We conclude this section with some examples of discontinuous isomorphisms of regular algebras, to show that the singuarity set F is not always empty. Another example will be given in Section 6. Discontinuous isomorphisms can always be constructed when there is a maximal ideal $\mathfrak{M}$ in $\mathfrak{A}$ such that $\mathfrak{M}^2$ is not closed in $\mathfrak{A}$. It is an open question whether such isomorphisms can be constructed in algebras such as $C(\Omega)$ or the group algebra of a locally compact abelian group (cf. [1]) where $\mathfrak{M}^2 = \mathfrak{M}$ for every regular maximal ideal.

Example 1. Let $\mathfrak{A}$ be the algebra l_p, $1 \leqq p < \infty$, under pointwise multiplication. (There is no unit, but we can adjoin one and consider l_p as a maximal ideal in $\mathfrak{A} \oplus \{\lambda e\}$.) It is easy to see that $(l_p)^2 = l_{p/2}$, which is a proper dense subset of l_p and thus cannot be closed. Let θ be a discontinuous linear functional in l_p which vanishes on $(l_p)^2$. Define $\mathfrak{B} = l_p \oplus \{\lambda r\}$, where $r(l_p) = 0 = r^2$, and define

$$\nu(x) = x + \theta(x) r, \qquad x \in l_p.$$

Then ν is a discontinuous isomorphism of l_p into $\mathfrak{B}$.

Example 2. Consider the algebra $\mathfrak{D}^1$ of continuously differentiable functions on $[0,1]$ with $\| x \| = \sup | x(t) | + \sup | x'(t) |$. The structure space is $[0,1]$. Define

$$\mathfrak{M} = \{x \mid x(0) = 0\}, \qquad \mathfrak{N} = \{x \mid x(0) = x'(0) = 0\}.$$

If $y \in \mathfrak{N}$, and p_n is any sequence of polynomials in $\mathfrak{M}$ converging uniformly to y', then the polynomials $q_n(t) = \int_0^t p_n(s) ds$ lie in $\mathfrak{M}^2$ and converge to y in $\mathfrak{D}^1$. Thus $\mathfrak{M}^2$ is dense in $\mathfrak{N}$. However, $\mathfrak{M}^2$ is not closed since it is easily seen that $y''(0)$ exists for every y in $\mathfrak{M}^2$. We may now use the argument of the last example to construct a discontinuous isomorphism of $\mathfrak{M}$ (and hence of $\mathfrak{D}^1$) into a Banach algebra.

4. Homomorphisms of $C(\Omega)$. In this section we consider the case that $\mathfrak{A}$ is the algebra $C(\Omega)$ of all continuous real or complex functions on a compact Hausdorff space Ω. Since $\mathfrak{M}^2 = \mathfrak{M}$ for every maximal ideal, the techniques for constructing discontinuous homomorphisms of the last section fail. In fact there are no known examples of discontinuous homomorphisms of $C(\Omega)$ for any Ω. In this section we shall strengthen the main result (Theorem 3.7) of the last section and exhibit the precise role the radical of the image must play if the homomorphism is to be discontinuous. We shall see that in the case of $C(\Omega)$ there is a decomposition of the homomorphism into the sum of two mappings, one continuous, and the other mapping into the radical of $\mathfrak{B}$. If the radical of the image is nil, the "radical" part of the homomorphism ν must be trivial, forcing ν to be continuous.

As before we denote by $F = \{\omega_1, \cdots, \omega_n\}$ the finite singularity set for ν: $C(\Omega) \to \mathfrak{B}$. It is convenient to introduce the following classes of functions:

1. $\mathfrak{M}(F)$ is the intersection of the n maximal ideals $\mathfrak{M}(\omega_i)$, $\omega_i \in F$.

2. $\mathfrak{J}(F)$ is the ideal of functions each of which vanishes in a neighborhood of F, the neighborhood depending on the function.

3. $\mathfrak{K}(F)$ is the dense subalgebra of $C(\Omega)$ consisting of those functions f such that $f(\omega) \equiv f(\omega_i)$ in a neighborhood of each point $\omega_i \in F$, the neighborhoods varying with f.

In the algebra $C(\Omega)$ relative units may always be chosen to have norm one. This fact allows a significant strengthening of Theorem 3.7.

4.1. THEOREM. *Let ν be a homomorphism of $C(\Omega)$ into a Banach algebra. If F denotes the singularity set of ν, then ν is continuous on the dense subalgebra $\mathfrak{K}(F)$ of $C(\Omega)$.*

Proof. It follows from Theorem 3.7 and the remark above that there is a constant M such that

$$\| \nu(g) \| \leqq M \| g \|, \qquad g \in \mathfrak{J}(F).$$

We now select functions e_i, $0 \leqq e_i \leqq 1$, such that $e_i e_j = 0$, $i \neq j$, and $e_i(\omega) \equiv 1$ in a neighborhood of $\omega_i \in F$. Then for any $f \in \mathfrak{K}(F)$, $f - \sum_{i=1}^{n} f(\omega_i) e_i \in \mathfrak{J}(F)$, so

$$\begin{aligned} \| \nu(f) \| &\leqq \| \nu(f - \sum_{i=1}^{n} f(\omega_i) e_i) \| + \| \sum_{i=1}^{n} f(\omega_i) \nu(e_i) \| \\ &\leqq M \| f - \sum_{i=1}^{n} f(\omega_i) e_i \| + \| f \| \sum_{i=1}^{n} \| \nu(e_i) \|. \\ &\leqq [(n+1)M + \sum_{i=1}^{n} \| \nu(e_i) \|] \| f \|, \qquad f \in \mathfrak{K}(F). \end{aligned}$$

Since ν is continuous on $\mathfrak{K}(F)$, it has a unique continuous extension to all of $C(\Omega)$,

4.2. *Definition.* We denote by μ the unique continuous homomorphism of $C(\Omega)$ into $\mathfrak{B}$ which agrees with ν on the dense subalgebra $\mathfrak{K}(F)$. Define

$$\lambda(f) = \nu(f) - \mu(f), \qquad f \in C(\Omega).$$

The mappings μ and λ will be called the *continuous* and *singular* parts of ν. We reserve the letter M now for a constant such that

$$\| \mu(f) \| \leqq M \| f \|, \qquad f \in C(\Omega).$$

The next theorem describes the structure of an arbitrary homomorphism of $C(\Omega)$.

4.3. THEOREM. *Let ν be a homomorphism of $C(\Omega)$ into a commutative Banach algebra $\mathfrak{B}$ and let $\mathfrak{R}$ be the radical of $\overline{\nu(C(\Omega))}$. Let $F = \{\omega_1, \cdots, \omega_n\}$ be the singularity set for ν and μ and λ be the continuous and singular parts of ν. Then:*

(a) *The range of μ is closed in $\mathfrak{B}$ and*

$$\overline{\nu(C(\Omega))} = \mu(C(\Omega)) \oplus \mathfrak{R},$$

the direct sum being topological.

(b) $\mathfrak{R} = \overline{\lambda(C(\Omega))}$.

(c) $\mathfrak{R} \cdot \mu(\mathfrak{M}(F)) = 0$, *and the restriction of* λ *to* $\mathfrak{M}(F)$ *is a homomorphism.*

(d) *There exist linear transformations* λ_i, $i = 1, \cdots, n$, *such that*

(i) $\lambda = \sum_{i=1}^{n} \lambda_i$,

(ii) $\mathfrak{R} = \mathfrak{R}_1 \oplus \cdots \oplus \mathfrak{R}_n$,

where $\mathfrak{R}_i = \overline{\lambda_i(C(\Omega))}$, *the direct sum being topological.*

(iii) $\mathfrak{R}_i \cdot \mathfrak{R}_j = 0$, $i \neq j$, *and* $\mathfrak{R}_i \cdot \mu(\mathfrak{M}(\omega_i)) = 0$,

$i = 1, \cdots, n$.

(iv) *The restriction of* λ_i *to* $\mathfrak{M}(\omega_i)$ *is a homomorphism.*

Proof. Let $\mathfrak{K} = \{x \mid \mu(x) = 0\}$. Since μ is continuous, $\mathfrak{K}$ is a closed ideal in $C(\Omega)$, so there exists a closed set $G \subseteq \Omega$ such that

$$\mathfrak{K} = \{x \mid x(\omega) = 0, \omega \in G\}.$$

If we give $C(\Omega)/\mathfrak{K}$ the norm

$$\| x + \mathfrak{K} \| = \inf\{\| y \| \mid y \in [x + \mathfrak{K}]\},$$

then by a theorem of Stone [10, Theorem 85], $C(\Omega)/\mathfrak{K}$ is isometrically isomorphic with $C(G)$, and

$$\| x + \mathfrak{K} \| = \sup_{\omega \in G} | x(\omega) |.$$

On the other hand, the semi-norm $| x | = \| \mu(x) \|$ is constant on the cosets $[x + \mathfrak{K}]$, so we may norm $C(\Omega)/\mathfrak{K}$ by defining $| x + \mathfrak{K} | = | x |$. By a theorem of Kaplansky [5] $| x + \mathfrak{K} | \geqq \| x + \mathfrak{K} \|$, $x \in C(\Omega)$. However, for any $y \in [x + \mathfrak{K}]$, we have $| x | = | y | \leqq M \| y \|$ since μ is continuous. Thus

$$| x + \mathfrak{K} | \leqq M \inf\{\| y \| \mid x - y \in \mathfrak{K}\} = M \| x + \mathfrak{K} \|,$$

showing the two norms are equivalent on $C(\Omega)/\mathfrak{K}$. To show μ has a closed range, suppose $b_0 \in \mathfrak{B}$ and $b_0 = \lim \mu(x_n)$. Then

$$\| x_m - x_n + \mathfrak{K} \| \leqq | x_m - x_n | = \| \mu(x_m - x_n) \| \to 0.$$

There exists $x_0 \in C(\Omega)$ such that $\| x_0 - x_n + \mathfrak{K} \| \to 0$. Thus

$$\| \mu(x_0) - \mu(x_n) \| \leqq M \| x_0 - x_n + \mathfrak{K} \| \to 0,$$

showing $b_0 = \mu(x_0)$. In particular, $\mu(C(\Omega))$ is algebraically and topologically isomorphic with $C(G)$. Thus, necessarily, $\mu(C(\Omega)) \cap \mathfrak{R} = (0)$.

We next prove that $\lambda = \nu - \mu$ maps into $\mathfrak{R}$. If $\phi \in \Phi_{\mathfrak{B}_1}$, $\mathfrak{B}_1 = \overline{\nu(C(\Omega))}$, then the functionals ϕ_ν and ϕ_μ on $C(\Omega)$ defined by $\phi_\nu(x) = \phi(\nu(x))$, $\phi_\mu(x) = \phi(\mu(x))$ are multiplicative, and hence continuous. Since they coincide on the dense subalgebra $\mathfrak{K}(F)$, we have $\phi_\nu = \phi_\mu$. Thus $\phi(\lambda(x)) = 0$, $x \in C(\Omega)$, $\phi \in \Phi_{\mathfrak{B}_1}$.

It follows now that $\nu(C(\Omega)) \subseteq \mu(C(\Omega)) \oplus \mathfrak{R}$. To complete the proof of (a) it suffices to show $\overline{\nu(C(\Omega))} = \mu(C(\Omega)) \oplus \mathfrak{R}$ since the algebraic direct sum must be topological as both of the factors are closed [6]. If $b = \lim \nu(x_n)$, then since $\mu(C(\Omega))$ is closed in $\mathfrak{B}$,[4]

$$\begin{aligned} \| \nu(x_m - x_n) \| &\geqq \rho_{\mathfrak{B}}(\nu(x_m - x_n)) \\ &= \rho_{\mathfrak{B}}(\mu(x_m - x_n)) \\ &= \rho_{\mu(C(\Omega))}(\mu(x_m - x_n)) \\ &\geqq M^{-1} \| \mu(x_m - x_n) \|. \end{aligned}$$

Thus there exists $x_0 \in C(\Omega)$ such that $\mu(x_0) = \lim \mu(x_n)$. If $r = b - \mu(x_0)$, then $r = \lim \lambda(x_n)$, so $r \in \mathfrak{R}$. This completes the proof of (a). Statement (b) follows from (a) and the last argument.

We next prove $\mathfrak{R} \cdot \mu(\mathfrak{M}(F)) = 0$. Since $\mathfrak{J}(F)$ is dense in $\mathfrak{M}(F)$, it is enough to prove that $\mu(z)\lambda(x) = 0$, $z \in \mathfrak{J}(F)$, $x \in C(\Omega)$. Now $xz \in \mathfrak{J}(F)$ and ν and μ agree on $\mathfrak{J}(F)$. Thus

$$\begin{aligned} \mu(z)\lambda(x) &= \mu(z)[\nu(x) - \mu(x)] \\ &= \nu(z)\nu(x) - \mu(z)\mu(x) \\ &= \nu(zx) - \mu(zx) = 0. \end{aligned}$$

If $x, y \in \mathfrak{M}(F)$, we have

$$\begin{aligned} \lambda(xy) &= \nu(xy) - \mu(xy) \\ &= [\mu(x) + \lambda(x)][\mu(y) + \lambda(y)] - \mu(xy) \\ &= \mu(x)\mu(y) + \lambda(x)\lambda(y) - \mu(xy) \\ &= \lambda(x)\lambda(y) \end{aligned}$$

since the cross product terms vanish. Thus $\lambda \colon \mathfrak{M}(F) \to R$ is a homomorphism and (c) is proved.

For (d) select functions e_i, $i = 1, \cdots, n$, such that e_i is one in a neighborhood of ω_i and $e_i e_j = 0$, $i \neq j$. Define

$$\lambda_i(x) = \lambda(e_i x), \qquad x \in C(\Omega).$$

[4] $\rho_{\mathfrak{A}}(y)$ denotes the spectral radius of y in the algebra $\mathfrak{A}$.

If $x, y \in \mathfrak{M}(\omega_i)$, then $e_i x, e_i y \in \mathfrak{M}(F)$, so

$$\lambda_i(x)\lambda_i(y) - \lambda_i(xy) = \lambda((e_i^2 - e_i)xy) = 0,$$

as $(e_i^2 - e_i)xy \in \mathfrak{J}(F)$. Thus $\lambda_i \colon \mathfrak{M}(\omega_i) \to \mathfrak{R}$ is a homomorphism. That $\mathfrak{R}_i \cdot \mathfrak{R}_j = 0$ is immediate. The relation $\lambda = \sum \lambda_i$ follows from the fact $(1 - \sum e_i)x \in \mathfrak{J}(F)$ for all $x \in C(\Omega)$.

All that remains is to prove (d) (ii) and the fact $\mathfrak{R}_i \cdot \mu(\mathfrak{M}(\omega_i)) = 0$. For these we need the relations

$$(*) \qquad \mu(e_i)\lambda(e_i y) = \lambda(e_i y), \quad y \in C(\Omega),$$

$$(**) \qquad \mu(e_j)\lambda(e_i y) = 0, \quad i \neq j, \quad y \in C(\Omega).$$

For (**) note

$$0 = \nu(e_i)\nu(e_j y) = \mu(e_i)[\mu(e_j y) + \lambda(e_j y)]$$
$$= \mu(e_i)\lambda(e_j).$$

For (*)

$$\lambda(e_i y) = [\mu(1 - \sum_{j=1}^{n} e_j) + \sum_{j=1}^{n} \mu(e_j)]\lambda(e_i y)$$
$$= \mu(e_i)\lambda(e_i y)$$

by (**) and (c). Now (d) (ii) follows directly. Finally, if $z \in \mathfrak{M}(\omega_i)$,

$$z = \sum e_j z + (1 - \sum e_j)z,$$

the last term being in $\mathfrak{J}(F)$. Thus by (**) and the fact $e_i z \in \mathfrak{M}(F)$, we have

$$\lambda(e_i x)\mu(z) = \lambda(e_i x) \sum_{j=1}^{n} \mu(e_j)\mu(z)$$
$$= \lambda(e_i x)\mu(e_i z) = 0.$$

It is an open problem whether $C(\Omega)$ admits a discontinuous homomorphism. The last theorem allows us to reduce this question to the question of the existence of a homomorphism of a maximal ideal of $C(\Omega)$ into a radical algebra. We summarize the situation in

4.4. THEOREM. *If the algebra $C(\Omega)$ has any one of the following, it has every other.*

(1) *An incomplete multiplicative norm,*

(2) *A discontinuous multiplicative semi-norm,*

(3) *A discontinuous isomorphism into a Banach algebra,*

(4) *A discontinuous homomorphism into a Banach algebra,*

(5) *A homomorphism* λ *into a radical Banach algebra with adjoined unit* $\mathfrak{R} \oplus \{\alpha e\}$, *such that for some maximal ideal* ω_0, $\lambda(\mathfrak{M}(\omega_0)) \subseteq \mathfrak{R}$ *and* $\lambda(\mathfrak{J}(\omega_0)) = 0$.

Proof. We know (1) $\Leftrightarrow$ (3) and (2) $\Leftrightarrow$ (4). Also (3) $\Rightarrow$ (4) $\Rightarrow$ (5), for we may take for λ one of the homomorphisms λ_i of the last theorem. Given (5), the norm $|x| = \|x\| + \|\lambda(x)\|$ defines a multiplicative norm on $\mathfrak{M}(\omega_0)$. This we can raise to $C(\Omega)$, showing (5) $\Rightarrow$ (1).

The next theorem gives a sufficient condition for a homomorphism of $C(\Omega)$ to be continuous.

4.5. THEOREM. *Let* ν *be a homomorphism of* $C(\Omega)$ *into a commutative Banach algebra* $\mathfrak{B}$. *If the radical* $\mathfrak{R}$ *of* $\mathfrak{B}$ *is a nil ideal* (*i.e.* $r^k = 0$ *for some* k *if* $r \in \mathfrak{R}$), *then* ν *is continuous.*

Proof. In view of the splitting $\nu = \mu + \lambda$ it suffices to prove $\lambda \equiv 0$ on $\mathfrak{M}(F)$, since for any $x \in C(\Omega)$,

$$x = [x - \sum_{i=1}^{n} x(\omega_i) e_i] + \sum_{i=1}^{n} x(\omega_i) e_i,$$

where the e_i are orthogonal functions and e_i is identically one on a neighborhood of ω_i. Thus

$$\lambda(x) = \lambda(x - \sum_{i=1}^{n} x(\omega_i) e_i)$$

as the last term belongs to $\mathfrak{K}(F)$. It is enough to show $\lambda(x) = 0$ for $x \geqq 0$, $x \in \mathfrak{M}(F)$. Consider the following functions on the interval $0 < t < 1$:

$$f(t) = e^{-1/t}, \qquad f_n(t) = t^{-n} e^{-1/t}, \qquad g(t) = -[\ln t]^{-1}.$$

They all approach zero as $t \to 0$. The functions $f(x)$, $f_n(x)$, and $g(x)$ are in $\mathfrak{M}(F)$ and $f(x) = x^n f_n(x)$. Since $\lambda(x)^n = 0$ for some n, $\lambda(f(x)) = 0$. However, if $y = g(x)$, then $y \geqq 0$, $y \in \mathfrak{M}(F)$, and hence $\lambda(f(y)) = 0$. But $x = f(y)$. Thus $\lambda(x) = 0$.

5. Non normable algebras. Let $\mathfrak{A}$ be a semi-simple regular algebra with unit, and let $\mathfrak{B}$ be a commutative Banach algebra. For a homomorphism $\nu \colon \mathfrak{A} \to \mathfrak{B}$ with singularity set F we define the ideals $\mathfrak{M}(F)$ and $\mathfrak{J}(F)$ as in Section 4. For any point $\varphi_0 \in \Phi_{\mathfrak{A}}$ the ideal $\mathfrak{J}(\varphi_0)$ is the set of x in $\mathfrak{A}$ each of which vanishes in some neighborhood of φ_0. In this section we investigate the kernel of a *radical homomorphism* of $\mathfrak{A}$, that is, a homomorphism which maps $\mathfrak{M}(F)$ into the radical $\mathfrak{R}$ of $\mathfrak{B}$. It is easy to see that the kernel must contain $\mathfrak{J}(F)$, since if $x \in \mathfrak{J}(F)$, we can find $y \in \mathfrak{J}(F)$ such that $xy = x$. Thus

$$\nu(x) = \nu(x)\nu(y)^n, \qquad n = 1, 2, \cdots,$$

and the right side converges to zero as $\nu(y) \in \mathfrak{R}$. It will be shown that whenever $\overline{\mathfrak{J}(F)} \sim \mathfrak{J}(F)$ contains an orthogonal sequence ($f_m f_n = 0$, $m \neq n$), then the kernel includes elements of $\overline{\mathfrak{J}(F)}$ which are not in $\mathfrak{J}(F)$. This result is used to show that certain quotient algebras of $\mathfrak{A}$ are not normable.

For the first theorem we introduce a condition originally due to Ditkin (cf. [7, p. 86]).

5.1. *Definition.* The algebra $\mathfrak{A}$ satisfies the condition (D) at a point φ_0 in $\Phi_{\mathfrak{A}}$ if for each $x \in \overline{\mathfrak{J}(\varphi_0)}$, there exists a sequence $\{y_n\} \subseteq \mathfrak{J}(\varphi_0)$ such that $\lim_n xy_n = x$.

5.2. LEMMA. *Let ν: $\mathfrak{A} \to \mathfrak{B}$ be a radical homomorphism and $\{f_n\} \subseteq \mathfrak{J}(F)$, $f_m f_n = 0$, $m \neq n$. Then $\nu(f_n)^3 = 0$ for all large n. If $\mathfrak{A}$ satisfies the condition (D) at each point of F, then $\nu(f_n)^2 = 0$ for all large n.*

Proof. Suppose the theorem is false and that $\nu(f_n)^3 \neq 0$ for all n. By normalizing we can assume for convenience that $\| \nu(f_n) \| = \| \nu(f_n)^3 \|$. Choose $z_n \in \mathfrak{J}(F)$ such that

$$\| f_n - z_n \| < 1/n^3 \, \| f_n \|, \qquad n = 1, 2, \cdots,$$

and define $h_n = n f_n (f_n - z_n)$, $h = \sum_{n=1}^{\infty} h_n$. Then

$$f_n h = f_n h_n = n f_n^3 - n f_n^2 z_n, \qquad n = 1, 2, \cdots,$$

so

$$\nu(f_n)\nu(h) = n\nu(f_n)^3, \qquad n = 1, 2, \cdots,$$

which implies $\| \nu(h) \| \geqq n$, $n = 1, 2, \cdots$. This is the desired contradiction.

If the condition (D) holds at each point of F the argument can be strengthened. For suppose

$$\nu(f_n)^2 \neq 0, \quad \| \nu(f_n) \| = \| \nu(f_n)^2 \|, \qquad n = 1, 2, \cdots.$$

It is easy to see there exist functions $y_n \in \mathfrak{J}(F)$ such that

$$\| f_n - f_n y_n \| < 1/n^3.$$

Let $h_n = n(f_n - f_n y_n)$, $h = \sum h_n$. Then

$$\nu(h)\nu(f_n) = n\nu(f_n^2), \qquad n = 1, 2, \cdots,$$

so $\| \nu(h) \| \geqq n$, $n = 1, 2, \cdots$.

We next examine the case $\mathfrak{A} = C(\Omega)$. If ν is any homomorphism of $C(\Omega)$, then it follows from Theorem 4.1 that ν maps bounded orthogonal

sequences into bounded sequences, whenever the carriers of the members lie in disjoint open sets. We can now remove this last restriction.

5.3. COROLLARY. *Let ν be a homomorphism of $C(\Omega)$ into a Banach algebra and suppose $f_m f_n = 0$, $m \neq n$. Then*

$$\sup \| \nu(f_n) \| / \| f_n \| < \infty.$$

If ν is a radical homomorphism, then $\nu(f_n) = 0$ for all large n.

Proof. It suffices to establish the result for non-negative sequences since the sequences $\{f_n^+\}$ and $\{f_n^-\}$ are orthogonal, where $f_n^+ = f_n \wedge 0, f_n^- = -(f_n \vee 0)$. Thus we may suppose $f_n \geqq 0$. Clearly, all but finitely many elements of the sequence belong to $\mathfrak{M}(F)$, so we may assume $f_n \in \mathfrak{M}(F)$, $n = 1, 2, \cdots$. Let λ be the singular part of ν (cf. Section 4). Applying the second statement of Lemma 5.2 to the sequence $\{f_n^{\frac{1}{3}}\}$, we see $\lambda(f_n) = 0$ for all large n. (The fact that λ is a homomorphism only on $\mathfrak{M}(F)$ does not affect the argument). The result now follows directly.

5.4. THEOREM. *Let $\mathfrak{A}$ be a regular algebra and let $\varphi_0 \in \Phi_{\mathfrak{A}}$. If φ_0 is the limit of a sequence of distinct points $\{\varphi_n\} \subseteq \Phi_{\mathfrak{A}}$, then the algebra $\mathfrak{A}/\mathfrak{J}(\varphi_0)$ is not normable.*

Proof. Suppose $\mathfrak{A}/\mathfrak{J}(\varphi_0)$ is a normed algebra under some norm and let $\nu\colon \mathfrak{A} \to \mathfrak{A}/\mathfrak{J}(\varphi_0)$ be the natural homomorphism. Since $\mathfrak{J}(\varphi_0)$ is contained in the unique maximal ideal $\mathfrak{M}(\omega_0)$ in $\mathfrak{A}$, $\mathfrak{A}/\mathfrak{J}(\varphi_0)$ has a unique maximal ideal. Thus ν is a radical homomorphism of $\mathfrak{A}$ into the completion of $\mathfrak{A}/\mathfrak{J}(\varphi_0)$, whose kernel is precisely $\mathfrak{J}(\varphi_0)$. The desired contradiction will be obtained if we can show the kernel must be larger. Since $\varphi_0 = \lim \varphi_n$, we can find disjoint open sets E_n such that $\varphi_n \in E_n$, and functions $g_n \in \mathfrak{A}$ with $\mathrm{car}(g_n) \subseteq E_n$. Arrange these functions in an infinite matrix by defining, for example,

$$h_{ij} = g_m, \text{ where } m = 2^{i-1}(2j-1), \qquad i, j = 1, 2, \cdots.$$

Let

$$f_j = \sum_{i=1}^{\infty} \alpha_{ij} h_{ij}, \qquad j = 1, 2, \cdots,$$

where the constants α_{ij} are chosen to make the series converge in $\mathfrak{A}$. Then $f_j f_k = 0$, $j \neq k$, and $f_j \in \overline{\mathfrak{J}(\omega_0)} \sim \mathfrak{J}(\omega_0)$. It follows from Lemma 5.2 that $\nu(f_j^3) = 0$ for all sufficiently large j.

6. An example.

We conclude with a discussion of an example, due to C. Feldman [2], which will show that the main boundedness theorem of Section 2 can not be improved.

Let $\mathfrak{A}$ be the commutative Banach algebra which is the completion of the algebra $\mathfrak{A}_0$ of all finite sums

$$\sum_{i=1}^{n} \alpha_i e_i + \gamma r,$$

where α_i and γ are complex, e_i are mutually orthogonal idempotents, $r^2 = 0$, $e_i r = r e_i = 0$, and

$$\| \sum \alpha_i e_i + \gamma r \| = \max\{[\sum | \alpha_i |^2]^{\frac{1}{2}}, | \gamma - \sum \alpha_i |\}.$$

We refer to [2] for a proof that $\mathfrak{A}$ is a Banach algebra. Let $\mathfrak{R}$ be the one dimensional ideal generated by r. Then $\mathfrak{A}_0/\mathfrak{R}$ is isometrically isomorphic with the algebra of finite sequences $[\alpha_i]$ with the norm $[\sum | \alpha_i |^2]^{\frac{1}{2}}$, so $\mathfrak{A}/\mathfrak{R}$ is the algebra l_2 and $\mathfrak{R}$ is the radical of $\mathfrak{A}$. Feldman proved that there is no closed subalgebra of $\mathfrak{A}$ isomorphic to $\mathfrak{A}/\mathfrak{R}$. Thus a natural generalization of the Wedderburn principal theorem for finite dimensional algebras is false for $\mathfrak{A}$. We shall show, however, that there does exist a non-closed subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ such that $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{R}$. The map $\nu\colon l_2 \to \mathfrak{B}$ will be the discontinuous isomorphism of l_2 which we seek. We summarize the information we need in the following theorem.

6.1. THEOREM. (a) *The span of the idempotents is dense in* $\mathfrak{A}$.

(b) *There exists no closed subalgebra* $\mathfrak{B}'$ *of* $\mathfrak{A}$ *such that* $\mathfrak{A} = \mathfrak{B}' \oplus \mathfrak{R}$.

(c) *There exists a non-closed subalgebra* $\mathfrak{B}$ *of* $\mathfrak{A}$ *such that* $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{R}$.

Proof. Let I denote the algebraic span of the idempotents e_i of $\mathfrak{A}$. To prove (a) it is enough to show r is in $\bar{I}$. Since the topology of l_1 is stronger than that of l_2 one may find sequences $\xi_n = [\alpha_i^{(n)}]$ of non-negative numbers, each having only finitely many non-zero terms, such that the norm of ξ_n approaches one in l_1 and zero in l_2. Let $x_n = \sum \alpha_i^{(n)} e_i$. Then $x_n \in I$ and

$$\| r - x_n \| = \max\{[\sum | \alpha_i^{(n)} |^2]^{\frac{1}{2}}, | 1 - \sum \alpha_i^{(n)} |\} \to 0. \qquad \text{Thus } \mathfrak{A} = \bar{I}.$$

To prove (b) suppose there exists a closed subalgebra $\mathfrak{B}'$ such that $\mathfrak{A} = \mathfrak{B}' \oplus \mathfrak{R}$. Then $e_i \in \mathfrak{B}'$ for each i, for writing $e_i = b + \gamma r$, $b \in \mathfrak{B}'$, we see $e_i = e_i^2 = (b + \gamma r)^2 = b^2 = b + \gamma r$. Thus $b^2 = b = e_i$, so $\mathfrak{B}' \supseteq I$. Since $\mathfrak{B}'$ is assumed closed, we have $\mathfrak{B}' = \mathfrak{A}$ by (a).

Now $l_1 \subseteq l_2 = \mathfrak{A}/\mathfrak{R}$ and, using Zorn's lemma, we may construct a vector subspace V of l_2 such that $l_2 = l_1 \oplus V$. We construct an isomorphism ν of l_2 into $\mathfrak{A}$ as follows: For $\xi = [\alpha_i] \in l_1$ define

$$\nu(\xi) = \sum \alpha_i e_i.$$

Since $\| e_i \| = 1$ in $\mathfrak{A}$ the series converges absolutely. For $\xi = [\alpha_i] \in V$ we have $\sum | \alpha_i | = +\infty$. Let

$$x_n = \sum_{i=1}^{n} \alpha_i e_i + \sum_{i=1}^{n} \alpha_i r.$$

Then $\{x_n\}$ is a Cauchy sequence in $\mathfrak{A}$ since

$$\| x_m - x_n \| = [\sum_{n+1}^{m} | \alpha_i |^2]^{\frac{1}{2}} \to 0.$$

Define $\nu(\xi) = \lim x_n$ in $\mathfrak{A}$. One shows easily that $(\rho\nu)(\xi) = \xi$, $\xi \in l_2$, where ρ is the Gelfand homomorphism. Let $\mathfrak{C}$ and $\mathfrak{D}$ denote the range of ν on l_1 and V respectively. Then $\mathfrak{C}$ is clearly an algebra. If $\mathfrak{B} = \mathfrak{C} \oplus \mathfrak{D}$ is an algebra, it follows that ν is an isomorphism and $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{R}$. If $x, y \in \mathfrak{D}$ and $x = \lim x_n$, $y = \lim y_n$, where

$$x_n = \sum_{i=1}^{n} \alpha_i e_i + \sum_{i=1}^{n} \alpha_i r, \qquad y_n = \sum_{i=1}^{n} \beta_i e_i + \sum_{i=1}^{n} \beta_i r,$$

then $x_n y_n = \sum_{i=1}^{n} \alpha_i \beta_i e_i$ and $[\alpha_i \beta_i] \in l_1$. Thus $xy \in \mathfrak{C}$. Similarly, if $x \in \mathfrak{D}$, $y \in \mathfrak{C}$, then $xy \in \mathfrak{C}$.

6.2. Corollary. *Let $\nu \colon l_2 \to \mathfrak{B}$ be the discontinuous isomorphism of the last theorem. If p_n is the idempotent $[1, 1, \cdots, 1, 0, 0, \cdots]$ in l_2 whose first n entries are ones, then $\| p_n \| = n^{\frac{1}{2}}$ in l_2, while $\| \nu(p_n) \| = n$ in $\mathfrak{A}$.*

The algebra $\mathfrak{A}$ has an additional interesting property.

6.3. Corollary. *The algebra $\mathfrak{A}$ admits two inequivalent complete multiplicative norms.*

Proof. Besides the given norm in $\mathfrak{A}$ we have the norm

$$||| x ||| = || y ||_2 + | \lambda |,$$

where $x = y + \lambda r$, $y \in \mathfrak{B}$, and $\| y \|_2$ is the norm of $\nu^{-1}(y)$ in l_2. The two norms are inequivalent since $\mathfrak{B}$ is not closed in $\mathfrak{A}$.

Added in proof. In Section 4 it was shown that when $\mathfrak{A} = C(\Omega)$, then $\nu = \mu + \lambda$, where μ is a continuous homomorphism and λ maps into the radical of the image algebra. One easily sees that this splitting of ν holds for an algebra $\mathfrak{A}$ whenever the following two conditions hold: (1) There are no closed non maximal primary ideals, i. e. $\mathfrak{M}(\phi) = \overline{\mathfrak{J}(\phi)}$, $\phi \in \Phi_{\mathfrak{A}}$. (2) There is a constant K such that if $\phi \in \Phi_{\mathfrak{A}}$, $g \in \mathfrak{J}(\phi)$, then there exists $h \in \mathfrak{J}(\phi)$ with $gh = g$, $\| h \| \leqq K$. Y. Katznelson has pointed out to us that the algebra $\mathfrak{F}$ of absolutely convergent Fourier series has these properties (cf. N. Wiener,

The Fourier integral and certain of its applications, Cambridge, 1933, page 88). Statements (c) and (d) of Theorem 4.3 carry over. However $\mu(\mathfrak{F})$ will contain radical elements whenever the ideal $\{x \mid \mu(x) = 0\}$ is not the kernel of its hull. The question of whether $\mathfrak{F}$ has discontinuous homomorphisms is thus reduced to the open question of the existence of a homomorphism of $\mathfrak{F}$ into a radical Banach algebra. Any such homomorphism is necessarily discontinuous.

One might think that condition (2) would imply (1). However Katznelson has constructed the following elegant example. Let $\mathfrak{A}$ be the algebra of all complex sequences $x = [\xi_0, \xi_1, \cdots]$ such that $\xi_n \to 0$ and $\sup_n n^{-\frac{1}{2}} \sum_{i=1}^{n} | \xi_i - \xi_{i-1} | < \infty$. The norm of x is the sum of this supremum and $\sup_n | \xi_n |$. Adjoin a unit obtaining $\mathfrak{A}_1$. Then $\Phi_{\mathfrak{A}_1}$ is the integers with point at infinity and $\{x \mid \lim_{n \to \infty} n^{-\frac{1}{2}} \sum_{i=1}^{n} | \xi_i - \xi_{i-1} | = 0\}$ is a closed primary ideal. But the idempotents $k_{[0,n]}$ form a bounded system of relative units.

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ON HYPERSURFACES WITH NO NEGATIVE SECTIONAL CURVATURES.*

By Richard Sackstedder.[1]

1. Introduction. If a hypersurface $S \subset E^{n+1}$ is the boundary of a sufficiently smooth convex body, S has the intrinsic properties that all of its sectional curvatures are non-negative and as a Riemannian manifold it is complete in the sense of Hopf and Rinow. Most of this paper is concerned with the converse question, that is, given a complete Riemannian manifold with no negative sectional curvatures immersed in E^{n+1}, when is the image the boundary of a convex body?

The second appendix is independent of the body of the paper. In it, some counterexamples are given which show that theorems of Hilbert and Weyl on the extremes of the curvatures of a surface are false without suitable smoothness assumptions.

2. Preliminaries. A Riemannian manifold is said to be of *class* C^k $(k \geqq 1)$ if it is of class C^k as a differentiable manifold and in any coordinate system the components of the metric tensor are functions of class C^{k-1}. Unless the contrary is stated, a manifold will mean a connected manifold without a boundary. Let M_1 and M_2 be manifolds of class C^k and of dimensions m_1 and m_2 $(m_1 < m_2)$ respectively. M_1 will be said to be C^m*-immersed* $(m \leqq k)$ in M_2 if there is a single-valued map $X: M_1 \rightarrow M_2$ of class C^m with Jacobian of rank m_1 at every point of M_1. Such an immersion will be called *isometric* if M_1 and M_2 are Riemannian manifolds and the metric induced on

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the image $X(M_1)$ as a subset of M_2 is the same as the metric induced locally from the metric of M_1. If the map F is one-to-one, M_1 will be said to be C^m-*imbedded* in M_2. In the special case in which M_1 is of dimension m and $M_2 = E^{m+1}$, the image $X(M_1)$ under an immersion will be called an *m-hypersurface*, of if $m = 2$, a *surface*.

A Riemannian manifold becomes a metric space if the distance between two points is defined to be the greatest lower bound of lengths of the arcs connecting them. A manifold[2] is said to be *complete* if the metric space obtained in this way is complete. A number of equivalent definitions of completeness will be used below, cf. [11] or [16]. The image of a complete n-manifold isometrically immersed in E^{n+1} is called a *complete n-hypersurface.*

The main result of this paper is the following theorem which generalizes a classical theorem of Hadamard [7].

THEOREM (*). *Let M be a complete Riemannian n-manifold* ($n \geqq 2$) *and $X: M \to E^{n+1}$ a C^{n+1} isometric immersion of M into E^{n+1}. Suppose that every sectional curvature of M is non-negative and that at least one is positive. Then the image $X(M)$ is the boundary of a convex body in E^{n+1}.*

For sufficiently smooth hypersurfaces this theorem generalizes results of Hadamard [7], Stoker [22], Van Heijenoort [23], and Chern and Lashof [4]. Hadamard's result corresponds to the case where M is compact and all sectional curvatures of M are positive. Stoker generalized Hadamard's result for $n = 2$ by removing the restriction that M is compact. Chern and Lashof obtained another generalization for $n = 2$ in which M is assumed to be compact but the sectional curvatures are only required to be non-negative rather than strictly positive. Finally, Van Heijenoort considered all $n \geqq 2$, but replaced the assumptions on sectional curvatures by local convexity conditions.

The condition that sectional curvatures of M be non-negative is easily seen to be equivalent to the condition that the second fundamental form of $X(M)$ be semi-definite; cf. Appendix 1. However, the non-negativeness of the sectional curvatures does not imply that the second fundamental form remains either non-negative or non-positive at all points and consequently does not imply any sort of local convexity property. This assertion is clear for hypercylinders. A less trivial example is given by the surface in the (x, y, z) space E^3 defined by $z = x^3(1 + y^2)$ for $|y| < \frac{1}{2}$. The second fundamental form of this surface is positive definite for $x > 0$ and negative definite for $x < 0$. It

[2] Hopf and Rinow [11] require that the manifold satisfy the second axiom of countability, but this condition is automatically fulfilled in a connected Riemannian manifold; cf. [20], p. 23.

follows from Theorem (*) that no neighborhood of the origin on this surface can be a part of a complete surface with non-negative Gaussian curvature.

In Theorem (*), the purpose of the assumption that at least one sectional curvature is positive is to eliminate the possibility that $X(M)$ is a non-convex hypercylinder. Consequently, Thsorem (*) complements a result of Hartman and Nirenberg [8], p. 912 who proved that if all sectional curvatures of M are zero, $X(M)$ is a hypercylinder.

Let r denote the maximum rank of the second fundamental form of $X(M)$. In Appendix 1, it is shown that, under the hypotheses of Theorem (*), r is determined intrinsically. Certain supplementary facts about $X(M)$ which follow from Appendix 1 and the proof of Theorem (*) are formulated below.

SUPPLEMENT TO THEOREM (*). *Under the hypotheses of Theorem (*), E^{n+1} can be decomposed as a product $E^{n+1} = E^{r+1} \times E^{n-r}$ in such a way that $X(M) = P_1X(M) \times P_2X(M)$, where P_1 and P_2 denote respectively the orthogonal projections onto E^{r+1} and E^{n-r}. Then $P_2X(M) = E^{n-r}$ and $P_1X(M)$ is a hypersurface in E^{r+1} which bounds a convex body which contains no complete line. The integer r is determined intrinsically and satisfies $2 \leqq r \leqq n$.*

The proof of Theorem (*) depends on a series of lemmas and theorems which are proved in the next two sections. The lemmas of Section 3 are general propositions, some of which are known or are implicit in the literature. No attempt is made to state these lemmas in the most general form in which they are valid, because to do so would make the applications of them less clear. The lemmas of Section 4 are more specifically related to the problem of this paper. Most of these lemmas are not of interest in themselves because the truth of their assertions is clear once Theorem (*) has been proved. Two propositions which are perhaps of some interest are formulated as Theorems 1 and 2.

3. Lemmas. Throughout this section M will denote a Riemannian manifold of class C^1 and $X: M \rightarrow E^{n+1}$ a C^1 immersion of M into E^{n+1}.

The boundary of a set S will be denoted by S'.

Suppose that V is an $(n+1)$-vector. The following definitions and notation will often be used in this and the next section. W will denote an open connected subset of M such that the inner product $N(p) \cdot V \neq 0$ at any point p of W. Here $N: M \rightarrow S^{n+1}$ is the "composition" of X and the normal map of $X(M)$. It may not be possible to define N in the large on M, but $N(p) \cdot V \neq 0$ has an obvious meaning. Let Π_V denote the hyperplane through the origin orthogonal to V. Let $(x^0, \cdots, x^n)$ be coordinates in E^{n+1} such that

V is a unit vector along the x^0-axis and Π_V is the hyperplane $x^0 = 0$. For p in W, let $X(p) = (z(p), Y(p)) = (x^0(p), \cdots, x^n(p))$ where $z(p)$ is a scalar and $Y: W \to \Pi_V$.

A pair of points p, q of W will be said to *lie on a segment in* W if there is an arc γ in W whose image under Y is the line segment in Π_V joining $Y(p)$ to $Y(q)$. Since Y is a local homeomorphism, γ is unique as a set, and γ will be called *the segment connecting* p *to* q. A subset U of W will be called W-*convex* if every pair of points in U which lie on a segment in W lie on a segment in U. Every component of a W-convex set is W-convex and the intersection of any family of W-convex sets is W-convex. Note that the notion of W-convexity depends on the vector V as well as on W.

LEMMA 1. *Let U be a closed connected subset of W which is W-convex. Then $Y(U)$ is convex and Y is one-to-one on U.*

Proof. The proof is quite similar to proofs which have been given for a theorem of Tietze, cf. [12], p. 448, or [2], p. 56, ex. 22. Consequently it will only be sketched.

The proof depends on the following facts: 1) if p and q are points of U, there are points $p_0 = p, p_1, \cdots, p_k = q$ of U such that p_{j-1} and p_j lie on a segment in U for $j = 1, 2, \cdots, k$, 2) If p_0 and p_1, and p_1 and p_2 lie in a segment in U, then p_0 and p_2 lie on a segment in U. The proofs of 1) and 2) can be carried out as in [2] or [12] in spite of the fact that Y is only locally one-to-one on W. That Y is one-to-one on U follows from the convexity of $Y(U)$, the connectedness of U, and the fact that Y is a local homeomorphism. This completes the proof of Lemma 1.

Let U be a subset of W. By the *W-convex hull of U* is meant the smallest W-convex set containing U. Let $U = U^0$ and define $U^1, U^2, \cdots$ inductively by $U^k =$ the union of all segments in W connecting pairs of points of U^{k-1}. Clearly $U^0 \subset U^1 \subset U^2 \cdots$.

LEMMA 2. *The W-convex hull of U is $U^\infty \equiv \bigcup_{k=1}^{\infty} U^k$.*

The proof of Lemma 2 is quite simple and will be omitted.

LEMMA 3. *Let M be a simply connected manifold and let A and B be closed subsets of M. Then if two points x, y are connected in $M - A$ and in $M - B$, they are connected in $M - (A \cup B)$.*

Lemma 3 is a special case of a known theorem, cf. [25], p. 242, Theorem 9.2. It can also be verified directly using the fact that it holds for $M = E^2$.

COROLLARY 1. *Let M be a simply connected manifold and let D be an open connected subset. Then each component of $M - D$ contains just one component of the boundary of D.*

COROLLARY 2. *Let M be a simply connected manifold and D an open subset of M. Suppose that C_1 and C_2 are disjoint components of D. Then there is a component B of $M - D$ which separates C_1 and C_2 in M and which contains a component of the boundary of C_1 which also separates C_1 and C_2 in M.*

Corollaries 2 and 2 follow from Lemma 3 by exactly the arguments used in [17] to prove Theorems 14.5 and 14.3, respectively; cf. [25], p. 47 ff.

LEMMA 4. *Let M_1 be an open subset of a simply connected manifold M. Suppose that C is a component of M_1. Let C_0 be the component of $C \cup (M - M_1)$ containing C and let C^* be the union of C with all of the components of $M - M_1$ which intersect the closure of C. Then $C_0 = C^*$.*

Proof. Clearly $C_0 \supset C^*$. Let $\{U_a\}$ be the components of $M - C$. Then, if each of the sets $U_a \cap C_0$ is connected, $C_0 = C^*$. On the other hand if, for example, $U_1 \cap C_0$ is not connected, $U_1 \cap C_0 = A \cup B$ where A and B are closed, disjoint and non-void. Since U_a' is connected by Corollary 1, it can be supposed that $U_1' \subset A$. Then $C_0 = (C_0 - B) \cup B$ and $C_0 - B$, B are closed, disjoint, and non-void. This contradicts the connectedness of C_0 and completes the proof.

LEMMA 5. *Let M be a complete Riemannian n-manifold and $X: M \to E^{n+1}$ an isometric immersion of class C^1. Suppose that $B_1, B_2, \cdots$ is a sequence of subsets of M which have the properties:* (a) *$X(B_i)$ is convex,* (b) *$X \mid B_i$ is a homeomorphism,* (c) *$\limsup B_i$ is non-empty, say $p \in \limsup B_i$,* (d) *$L = \lim X(B_i)$ exists. Then there is a subset B of $\limsup B_i$ containing p which is such that $X \mid B$ is a homeomorphism and $X(B) = L$.*

Note: Here $\limsup B_i$ and $\lim X(B_i)$ are used in the sense of [26], p. 10.

Proof. By (c), it can be supposed that $B_1, B_2, \cdots$ is such that there is a sequence of points $p_1, p_2, \cdots$ satisfying $p_i \in B_i$ and $\lim p_i = p$ as $i \to \infty$. If $L \neq X(p)$, let $q \neq X(p)$ be a point of L, and let $q_1, q_2, \cdots$ be a sequence of points such that $q_i \in B_i$ and $q = \lim X(q_i)$ as $i \to \infty$. L is clearly convex, in particular, the segment $X(p)q$ connecting $X(p)$ to q is in L. Let coordinates $(x^0, x^1, \cdots, x^n)$ be chosen in E^{n+1} such that $X(p) = (0, \cdots, 0)$ and $q = (1, 0, \cdots, 0)$. Let $p_t = (t, 0, \cdots, 0)$. Then p_t is in L for $0 \leqq t \leqq 1$.

Let s denote the largest t value such that the segment p_0p_s, open at p_s, has a homeomorphic preimage containing p.

First it will be shown that $s > 0$. Let U be a compact neighborhood of p such that $X \mid U$ is a homeomorphism and let C_i be the component of $U \cap X^{-1}(p_iq_i)$ containing p_i. Then for large i, $X(C_i)$ is a non-degenerate closed segment. Let r_i be the point of C_i which maps into the endpoint of $X(C_i)$ which is not $X(p_i)$. Since U is a compact neighborhood of p it can be assumed that $\lim r_i = r$ exists as $i \to \infty$ and $r \neq p$. Since $X \mid U$ is a homeomorphism and $\lim X(C_i) = X(p)X(r)$ exists, it follows that $\lim C_i = C$ exists and $X(C)$ is the segment $X(p)X(r)$. Clearly this segment will be a subset of $X(p)q$. This proves that $s > 0$.

Now it will be proved that $s = 1$, i.e., that $p_s = q$. If $s \leqq 1$, the half open segment p_0p_s has a homeomorphic preimage in M. Since M is isometric and M is complete the closed segment p_0p_s has a homeomorphic preimage in M containing p. But now the argument just used to prove that $s > 0$ can be applied again to show that if $s < 1$, s can be increased. Consequently, $s = 1$, and $X(p)q$ has a homeomorphic preimage in M which contains p. The preimage is easily seen to be unique.

Let B be the union of all of the preimages corresponding to all of the points q in L. Then $X(B) = L$ and since B is connected and X is a local homeomorphism, $X \mid B$ is a homeomorphism. This completes the proof of Lemma 5.

4. Flat points on M. Let (H) denote the hypothesis: (i) M is a complete Riemannian n-manifold $(n \geqq 2)$ of class C^{n+1}, (ii) $X\colon M \to E^{n+1}$ is an isometric immersion of class C^{n+1}, (iii) all of the sectional curvatures of $X(M)$ are non-negative.

Let $h_{ij}(p)$ be the coefficients of the second fundamental form of $X(M)$ at the point p of M, relative to some fixed local coordinate system and choice of the unit normal vector $N\colon M \to S^n$. Let M_0, M_1 be the subsets of M defined as follows:

$$M_0 = \{p\colon p \in M, h_{ij}(p) = 0, 1 \leqq i, j \leqq n\}, \quad M_1 = M - M_0.$$

Points in M_0 will be called *flat points.*

LEMMA 6. *Assume (H). Let T be a connected subset of M_0. Then the normal N is constant on T and $X(T)$ lies in a hyperplane orthogonal to N on T.*

Proof. At a flat point of M the rank of the map $N: M \to S^n$ is zero. Since N is a map of class C^n a theorem of Sard [21], p. 888, implies that $N(M_0)$ is a one-dimensional zero set, in particular, $N(M_0)$ is totally disconnected. This proves that $N(T)$ consists of a single point, N_0. Let $(u^1, \cdots, u^n)$ be local coordinates near a point p_0 of T and let $f(u^1, \cdots, u^n) = N_0 \cdot (X(p) - X(p_0))$. Then $\partial f / \partial u^i = 0$ for $i = 1, \cdots, n$ on T, hence, by a theorem of A. P. Morse [15], f is constant on T. It follows that $N_0 \cdot X(p) = N_0 \cdot X(p_0)$ on T, that is $X(T)$ lies in the hyperplane $N_0 \cdot (x^1, \cdots, x^n) = N_0 \cdot X(p_0)$. This proves Lemma 6.

Remark. It is only for the purpose of proving Lemma 6 that the immersion in Theorem (*) is required to be of class C^{n+1}, instead of C^2. The non-negativeness of the sectional curvatures of M was not used in the proof of Lemma 6.

LEMMA 7. *Assume (H) and suppose that M is simply connected. Let T be a component of M_1. Let $W \subset M$ be as in Section 3. Then the intersection of each component of $M - T$ with W is W-convex.*

Proof. Let $\{U_a\}$ be the compents of $M - T$. Then by Corollary 1, U_a contains only one component of $T' = (M - T)'$, hence U_a' is connected. Lemma 6 implies that the normal to $X(M)$ is constant on U_a', say $N \equiv N_a$ on U_a' and there is a hyperplane Π_a with normal N_a such that $X(U_a') \subset \Pi_a$.

As in Section 3, let V be an $(n+1)$-vector such that for p in W, $N(p) \cdot V \neq 0$, let Π be a hyperplane in E^{n+1} orthogonal to V, and suppose that E^{n+1} has orthogonal coordinates $(x^0, \cdots, x^n)$ such that Π is $x^0 = 0$. If $N_a \cdot V \neq 0$, let $z(x^1, \cdots, x^n; a)$ denote the x^0 coordinate of the point on Π_a whose orthogonal projection onto Π is $(0, x^1, \cdots, x^n)$. Define the function $g: W \to E^1$ by $g(p) = x^0(p)$ if p is in $W \cap T$, $g(p) = z(Y(p); a)$ if p is in $W \cap U_a$, where $X(p) = (x^0(p), \cdots, x^n(p))$ and $Y(p) = (x^1(p), \cdots, x^n(p))$. Then g is of class C^2 by Lemma 6, since $U_a' \subset M_0$.

It can be supposed that the second fundamental form of $X(M)$ is non-negative semi-definite in T. Let a be fixed, let $J(p; a) = g(p) - z(Y(p); a)$, and define a (not necessarily isometric) immersion $G_a: W \to E^{n+1}$ by $G_a(p) = (J(p; a), Y(p))$. Then G_a is a C^2 immersion, the flat points of the immersion are precisely the points of $W - T$, and the second fundamental form of $G_a(W)$ is non-negative semi-definite. Let V_a be the W-convex hull of $U_a \cap W$. To prove that $U_a \cap W$ is W-convex it suffices to show that all points of V_a are flat points of $G_a(W)$, so that V_a does not meet T and, hence, $V_a = V_a \cap W$.

Let $U^0 = U_a \cap W$ and define $U^1, U^2, \cdots, U^\infty$ as in Lemma 2. Then

$V_a = U^\infty$ and to show that V_a consists entirely of flat points it suffices to prove that $U^0 = U^1$, hence $U^0 = U^\infty$. First it will be proved that

$$(1) \qquad J(p;a) \equiv 0 \text{ on } U^1.$$

Note that by definition of $J(p;a)$, $J(p;a) \equiv 0$ on U^0. If p_1, p_0 are points of U^0 which lie on a segment in W, let p_t denote the point on the segment which is such that $Y(p_t) = tY(p_1) + (1-t)Y(p_0)$, and let $H(t) = J(p_t;a)$ for $0 \leqq t \leqq 1$. $H''(t) \geqq 0$ because the second fundamental form of $G_a(W)$ is non-negative. $H(0) = H(1) = 0$ and $H'(0) = 0$ because p_1 and p_0 are in U^0. These statements imply $H(t) \equiv 0$ for $0 \leqq t \leqq 1$. This proves (1).

To complete the proof that $U^0 = U^1$, note that Y is one-to-one in a neighborhood S of the segment connecting p_1 to p_0. Then there is a function f defined in $Y(S)$ such that $J(p;a) \equiv f(Y(p))$ for p in S. It can be assumed that $Y(S)$ is convex. Then f will be a convex function. Let Δ^j denote the n-vector whose j-th component is Δ and other components are 0. Fix t, $0 < t < 1$ and put

$$Q(\Delta) = tf(Y(p_1) + \Delta^j) + (1-t)f(Y(p_0) + \Delta^j) - f(Y(p_t) + \Delta^j).$$

Q is defined for small Δ and by the convexity of f, $Q \geqq 0$. By (1), $Q(0) = 0$, so Q has a minimum at $\Delta = 0$. Therefore

$$Q''(0) = tf_{jj}(Y(p_1)) + (1-t)f_{jj}(Y(p_0)) - f_{jj}(Y(p_t)) \geqq 0.$$

Since p_1 and p_0 are in U^0, $f_{jj}(Y(p_1)) = f_{jj}(Y(p_0)) = 0$ and so $f_{jj}(Y(p_t)) \leqq 0$. On the other hand, $f_{jj}(Y(p_t)) \geqq 0$ because f is convex. Therefore $f_{jj}(Y(p_t)) = 0$ for $j = 1, \cdots, n$ and p_t is a flat point of $G_a(W)$, hence p_t is not in T and $U^0 = U^1$. This shows that V_a consists entirely of flat points of $G_a(W)$ and completes the proof of Lemma 7.

THEOREM 1. *Assume (H). Let C be a component of the set of flat points of M. Then $X(C)$ is convex and $X \mid C$ is a homeomorphism.*

Proof. Without loss of generality, M can be assumed to be simply connected, for otherwise M can be replaced by its universal covering manifold. By Lemma 6, N is constant on C. It can be supposed that $N \equiv V$ on C. Let W be the component of the set $\{p: p \in M, N(p) \cdot V \neq 0\}$ containing C. Let $\{T_a\}$ be the components of M_1 and let U_a be the component of $M - T_a$ containing C. Lemma 7 shows that $W \cap U_a$ is W-convex; hence the component of the intersection of all of these sets which contains C is W-convex. This component is a connected subset of M_0 hence it is C. This proves that C is W-convex. Lemma 1 shows that $Y(C)$ is convex, where $Y\colon W \to \Pi_V$ is as in

Section 3. V is the normal to $X(M)$ on C, hence $Y(C)$ is just a translation of $X(C)$. This proves that $X(C)$ is convex and since C is connected, $X \mid C$ is a homeomorphism. This completes the proof of Theorem 1.

LEMMA 8. *Assume (H) and suppose that M is simply connected. Let C_1 and C_2 be distinct components of M_1. Then there is a subset L of C_1' such that L separates C_1 from C_2 in M and $X(L)$ is an $(n-1)$-flat.*

Proof. By Corollary 2, there is a component B of M_0 which separates C_1 from C_2 and which contains a component L of C_1' which also separates C_1 from C_2. B is isometric to a convex subset of E^{n+1} by Theorem 1. By considering all possible types of convex subsets of dimension n or less (cf. [3], p. 3), it is easy to verify that B can separate a complete n-manifold only if B is isometric to an $(n-1)$-flat or to a set bounded by two parallel $(n-1)$-flats in E^n. (A set of the latter type will be called a *slab* below.) L is a connected subset of B' which separates C_1 from C_2. Clearly $X(L)$ must be an $(n-1)$-flat in either case.

LEMMA 9. *Assume (H) and suppose that M is simply connected. Suppose that C is a component of M_1 and that C_0 is the component of $M_0 \cup C$ containing C. Let p be a point of C_0'. Then there is a subset L of C' lying in the same component of M_0 as p which is such that $X(L)$ is an $(n-1)$-flat.*

Proof. Let B be the component of M_0 containing p. Let $p_1, p_2, \cdots$ be a sequence of points in $M_1 - C$ such that $p = \lim p_i$ as $i \to \infty$. By Lemma 8 there is a component B_i of C' which is such that $X(B_i)$ is an $(n-1)$-flat and B_i separates p_i from C. Either $B_j \subset B$ for some j or $B_i \cap B$ is empty for all i. In the first case, let $L = B_j$ and the lemma is proved for this case.

If $B_i \cap B$ is empty for all i, it will first be shown that $p \in \limsup B_i$. To see this, let q be a point of $B \cap C'$ (cf. Lemma 4). Let pq be the inverse image in B of the segment $X(p)X(q)$. Then B_i does not intersect pq for any i, hence B_i separates p from p_i. A simple application of Lemma 5 shows that there is a set $L \subset \limsup B_i \subset C'$ such that $X(L)$ is an $(n-1)$-flat. This completes the proof of Lemma 9.

LEMMA 10. *Assume the conditions of Lemma 9. Then there is a subset K of C_0 containing p such that $X(K)$ is an $(n-1)$-flat. In addition there is a neighborhood U of p with the properties:* (i) *$U - K$ has exactly two components D_1 and D_2,* (ii) *D_1 and D_2 are homeomorphic to n-cells.* (iii) *$D_1 \subset C_0$,* (iv) *$D_2 \cap C_0 = 0$. Finally, if Π is any hyperplane which is not parallel to $N(p)$, U can be chosen such that the orthogonal projection of*

$X(U \cap C_0)$ onto Π is a solid n-hemisphere (including the equatorial hyperplane).

Proof. Let B be the component of M_0 containing p. It follows from Theorem 1 and Lemma 9 that $X(B)$ is either an $(n-1)$-flat or a slab. In either case there is a subset $K \subset B \subset C_0$ such that p is in K and $X(K)$ is an $(n-1)$-flat. Let U be a neighborhood of p which is so small that $X \mid U$ is a homeomorphism and $X(U)$ has a homeomorphic orthogonal projection into a hyperplane $\Pi \subset E^{n+1}$. Let $Y: U \to \Pi$ denote the composition of X with the projection. It can be supposed that $Y(U)$ is an open convex subset of Π and in case $X(B)$ is a slab that U intersects only one component of B'.

Clearly U satisfies the conditions (i) and (ii) above. It remains to show that U can be chosen to satify (iii) and (iv). (The final assertion will be clear from the proof.) Consider the two cases $p \notin C'$ and $p \in C'$. In the first case, Lemma 9 shows that $X(B)$ is a slab. Then one of the components of $U-K$ will lie in $B \subset C_0$ because U only intersects one component of B'. Denote this component by D_1 and the other by D. We have shown that $D_1 \subset C_0$ if p is not in C'. If p is in C', denote by D_1 a component of $U-K$ which contains points of C arbitrarily close to p and let D be the other component. It will be shown that $D_1 \subset C_0$ in this case also, provided that U is small enough, i.e., it will be shown that there are no points in $D_1 - C_0$ arbitrarily close to p.

Suppose that there is a sequence of points $p_1, p_2, \cdots$ of $D_1 - C_0$ such that $p = \lim p_i$ as $i \to \infty$. It can be assumed that the points p_i are in $D_1 - C_0 - M_0$, because if all points of D_1 near p are in M_0 then all points of D_1 near p are in $B \subset C_0$. Lemma 8 implies that there are subsets $B_i \subset C'$ which separate p_i from C and are such that $X(B_i)$ is an $(n-1)$-flat. Then p_i and $C \cap U$ are in separate components of $U - B_i$. In fact, $U - B_i$ has exactly two components, one of which contains p_i while the other contains C. Also, $D \cup K^*$ is in the component of $U - B_i$ containing C, where $K^* \equiv K \cap U$. To verify this note that B_i intersects the connected set D_1 because B_i separates $p_i \in D_1$ from C and C intersects D_1 by definition of D_1. If B_i intersects D, B_i intersects K^* and $X(B)$ is a slab. Then p is an interior point of B, which contradicts $p \in C'$. Hence B_i does not intersect D which along with $B_i \cap D_1 \neq 0$ implies that B_i does not intersect K^*. Since p is in $C' \cap K$ this shows that $D \cup K^*$ is in the component of $U - B_i$ containing $C \cap U$.

Let F denote the intersection of all of the components of $U - B_i$ which contain $C \cap U$. Then $C \cap U \subset F$ and $D \cup K^* \subset F$. $p = \lim p_i$ implies that $Y(p)$ is a boundary point of the convex set $Y(F) \supset Y(D)$. This shows that

$F = D \cup K^* = U - D_1 \supset U \cap C$ which contradicts the definition of D_1. This proves that U can be chosen such that (iii) is satisfied.

To verify that U can be chosen such that (iv) is satisfied, first note that D cannot contain points of C arbitrarily close to p because then for a suitable choice of U, $D \subset C_0$ by the argument just completed and $U \subset C_0$ which contradicts $p \in C_0'$. Therefore suppose that U is so small that $D \cap C$ is empty. If (iv) does not hold for a suitable choice of U there are points of $D \cap C_0'$ arbitrarily close to p. Let $q \in D \cap C_0'$ and let S denote the interior of the convex hull of $Y(K^*) \cup Y(q)$ and let $D_2 = Y^{-1}(S)$. It will be shown that D_2 contains no points of C_0' and this will complete the proof of Lemma 10.

If there is a point r of $C_0' \cap D_2$, let B_r be the component of M_0 containing r. Lemma 9 shows that there is a subset L_r of $B_r \cap C'$ such that $X(L_r)$ is an $(n-1)$-flat. L_r does not intersect D because $D \cap C'$ is empty. Theorem 1 shows that $X(B_r)$ is a slab, hence $Y(B_r \cap U)$ is the intersection of $Y(U)$ with a closed half space of Π. B_r cannot intersect K, because this would imply that $K \subset B_r$, and that p is an interior point of B_r. This shows that q is in the interior of B_r because otherwise $Y(r)$ could not be in the interior of the convex hull of $Y(K^*) \cup Y(q)$. But if q is in the interior of B_r, q is not in C_0'. This contradiction proves Lemma 10.

LEMMA 11. *Assume (H). Let L be a subset of M such that $X(L)$ is k-flat, $0 < k < n$. Then the normal to $X(M)$ is constant on L.*

Proof. It is sufficient to prove the lemma for the case $k = 1$. Let p be on L and let E^{n+1} have coordinates $(x^0, \cdots, x^n)$ such that $X(p)$ is the origin, the unit normal to $X(M)$ is in the positive x^0-direction and $X(L)$ is the x^1 axis. Then, near p, $X(M)$ can be represented by $x^0 = z(x^1, \cdots, x^n)$. The matrix of second partial derivatives of z will be semi-definite and $z_{11} \equiv 0$ near p on L. But these conditions are easily seen to imply that $z_{1j} \equiv 0$ near p on L for $j = 1, \cdots, n$. This implies that the normal is constant on L near p, hence on all of L.

LEMMA 12. *Assume the conditions of Lemma 9. Let L be a subset of C_0, such that $X(L)$ is a k-flat $0 < k < n$. Then every point p of C_0 belongs to a subset $L_p \subset C_0$ such that $X(L_p)$ is a k-flat parallel to $X(L)$.*

Proof. It suffices to prove the case $k = 1$. Let C^* denote the set of points p which belong to a subset L_p of C_0 such that $X(L_p)$ is a 1-flat parallel to the 1-flat $X(L)$. Lemma 5 implies that C^* is closed. To see that C^* is open in C_0 it is sufficient to show that if $q \in L$, there is a neighborhood V of q such that $V \cap C_0 \subset C^*$.

It can be supposed that E^{n+1} has orthogonal coordinates $(x^0,\cdots,x^n)$ such that $X(q)$ is the origin, $N(q)=e$ is the unit vector in the positive x^0-direction, and $X(L)$ is the x^1-axis. Let Π denote the hyperplane $x^0=0$, let W be the component of $\{p: p\in M, N(p)\cdot e_0\neq 0\}$ containing L, and $W_0=W\cap C_0$. Such a component exists by Lemma 11. Define the map $Y: W\to\Pi$ by $Y(p)=(x^1(p),\cdots,x^n(p))$, where $X(p)=(x^0(p),\cdots,x^n(p))$. Let $V\subset W$ be a neighborhood of q such that $Y(V\cap C_0)$ is either a solid n-sphere or an n-hemisphere (including the equatorial hyperplane). Such a neighborhood exists by Lemma 10. Let p^* be an arbitrary point of $V\cap(C_0-L)$. It can be supposed that $Y(p^*)=(0,1,0,\cdots,0)$ and that for some $\epsilon>0$ there is a point $p\in V\cap(C_0-L)$ such that $Y(p)=(0,1+\epsilon,0,\cdots)$. Define q_t as the unique point of L such that $x^1(q_t)=t$ and let $T_t\subset\Pi$ be the solid closed triangle $Y(q)Y(p)Y(q_t)$. Define S_t to be the component of the inverse image $Y^{-1}(T_t)$ containing q, and let s, $0\leqq s\leqq+\infty$, be defined by $s=\sup\{t: Y(S_t\cap W_0)=T_t\}$. If $t<s$, $Y\mid S_t$ is a homeomorphism and if $t'\leqq t$, $S_{t'}\subset S_t$.

Now it will be proved that $s=+\infty$. Clearly $s>0$ because of the form of $V\cap C_0$. Suppose, if possible, that $s<+\infty$. Let $S^*=\cup\{S_t: 0<t<s\}$. First it will be shown that S^* is bounded, i.e. that the intrinsic distance between pairs of points of S^* is uniformly bounded. $X(S^*)$ can be represented in the form $x^0=z(x)$ for $x=(x^1,\cdots,x^n)$ in $Y(S^*)$. z is a convex (or concave) function of x on the interior of $Y(S^*)$. If z is convex, then $0\leqq z(x)\leqq z(Y(p))$ for x in $Y(S^*)$ because $z\equiv 0$ and $\partial z/\partial x^i\equiv 0$ on $Y(L)$. A similar inequality holds if z is concave. In either case, $|z(x)|\leqq|z(Y(p))|$ for x in $Y(S^*)$. The intrinsic distance between two points u_1 and u_2 of S^* satisfies the inequality

$$\begin{aligned}\operatorname{dist}(u_1,u_2)&\leqq\operatorname{dist}(Y(u_1),Y(u_2))+|z(Y(u_1))|+|z(Y(u_2))|\\&\leqq\operatorname{dist}(Y(p),Y(q_s))+2\,|z(Y(p))|.\end{aligned}$$

This proves that S^* is bounded. The completeness of M implies that every infinite subset of S^* has a limit point in M. Let S_0 be the closure of S^*. It is easy to verify that Y can be extended to homeomorphism of S_0 onto the closed triangle, $Y(q)Y(p)Y(q_s)$ using the results just proved.

Denote by pq_s the inverse image in S_0 of the segment $Y(p)Y(q_s)$. If the interior of the pq_s (i.e., the inverse image of the interior of the segment $Y(p)Y(q_s)$) is contained in the intersection of the interior of C_0 and W, then it is easy to verify that $Y(S_t\cap W_0)=T_t$ for some $t>s$. This is impossible by definition of s, hence the interior of pq_s intersects either W' or C_0'.

Let r be the point in the intersection of pq_s with $W' \cup C_0'$ such that the distance from $Y(p)$ to $Y(r)$ is as small as possible. Note that $r \neq p, q_s$.

There are two cases to consider, $r \in W'$ and $r \in W \cap C_0'$. If r is in W', let E^{n+1} have orthogonal coordinates $(y_0, \cdots, y_n)$ such that $N(r)$ is the unit vector in the positive y_0 direction and $X(r)$ is the origin. Note that $N(r)$ is orthogonal to the 2-flat determined by $Y(p)Y(r)$ and the unit vector e_0. To verify this, let u be the orthogonal projection of $N(r)$ onto the 2-flat. Then $N(r) \cdot e_0 = 0$ implies that $u \cdot e_0 = 0$, and it remains to verify $u \cdot Y(p)Y(r) = 0$. The image under X of pq_s is a convex curve in the 2-flat with orthogonal projection $Y(p)Y(q_s)$ on $\Pi \colon x^0 = 0$, and u is normal to the curve at $X(r)$. If $u \cdot Y(p)Y(r) \neq 0$, the tangent to the curve is in the e_0 direction at $X(r)$ and the curve has an inflection point at $X(r)$. Since convex curves do not have inflection points this is a contradiction. This proves that the 2-flat is a subset of the hyperplane $y_0 = 0$, in particular, the image of pr under X is in $y_0 = 0$.

By Lemma 10 there is neighborhood U of r such that the orthogonal projection of $X(U \cap C_0)$ onto $y_0 = 0$ is either a solid n-sphere or a hemisphere. Call the projected set R. It can be assumed that $X(U)$ is represented by $y_0 = w(y)$, $y = (y_1, \cdots, y_n)$ for y in R. The function w will be convex (or concave) in R and will satisfy $w = \partial w/\partial y^i = 0$ for $1 \leqq i \leqq n$ at $y = 0$, hence w does not change sign in R. Let v be any point of U on pr. Then $y_0(v) = 0$. Such a point v cannot be in the interior of C_0 because $N(v) \neq N(r)$ by definition of r, hence $y_0(v) = 0$ implies that y_0 changes sign near v, and w changes sign near the image of v. On the other hand, if there are no points of pr in the interior of C_0, r is in C_0' and all points u of pr near r are in the set K defined in Lemma 10. By Lemma 11, N is constant on K, hence $N(v) = N(r)$ for v in K. This contradicts the definition of r and proves that $r \in W'$ cannot occur.

Now consider the second possibility, that is, that r is in $W \cap C_0'$. Let U and K be as in the conclusion of Lemma 10. Then there is a subsegment of $Y(p)Y(q_s)$ centered at $Y(r)$ and contained in $Y(K \cap U)$. Since $U \cap K \subset U \cap C_0'$, this implies all points on pr near r are in C_0', which contradicts the definition of r. This completes the proof that $s = +\infty$.

It is clear from the proof that $s = +\infty$ that $\inf\{t \colon Y(S_t \cap W_0) = T_t\} = -\infty$. This shows that $Y(W_0)$ contains a strip

$$-\infty < x^1 < +\infty, \; 0 \leqq x^2 < 1 + \epsilon, \; x^i = 0 \text{ for } i > 2.$$

It can be supposed that z is a convex function on the strip, hence $z \geqq 0$ on the strip. Now it will be proved that $\partial z/\partial x^1 = 0$ along the line $x^2 = 1$, $x^i = 0$

for $i > 2$. Suppose if possible that $\partial z/\partial x^1 \neq 0$ at $x_t = (t, 1, 0, \cdots)$. Then since $z = 0$ on $Y(L)$, the convexity of z implies that for all v, $-\infty < v < +\infty$ $z(t+v, 0, 0, \cdots) = 0 \geqq z(x_t) + v\partial z/\partial x^1(x_t) + 1 \cdot \partial z/\partial x^2(x_t)$. Letting v approach $\pm\infty$ gives a contradiction. This shows that $\partial z/\partial x^1 = 0$ along the line $x^2 = 1$, $x^i = 0$ for $i > 2$, hence $z(t, 1, 0, \cdots)$ is constant for $-\infty < t < +\infty$. Let L^* denote the component of the inverse image of the line $x^2 = 1$, $x^i = 0$, $i > 2$ under Y which contains p^*. Then $X(L^*)$ is a 1-flat parallel to the x^1 axis. This completes the proof of Lemma 12 because p^* was an arbitrary points of C_0 in a neighborhood of q.

THEOREM 2. *Assume (H). Then either the set M_1 of non-flat points of M is connected or $X(M)$ is a hypercylinder.*

Proof. There is no loss of generality in assuming that M is simply connected, for otherwise M can be replaced by its universal covering manifold. Suppose that M_1 is not connected and let C be any component of M_1. Lemma 8 implies that there is a subset $L \subset C'$ such that $X(L)$ is an $(n-1)$-flat. Lemma 12 implies that every point p of C is in a set L_p such that $X(L_p)$ is a $(n-1)$flat. This shows that the rank of the second fundamental form of $X(M)$ is less than two at p, hence the maximum rank of the second fundamental form is one. Then Theorem III of Hartman and Nirenberg [8] shows that $X(M)$ is a hypercylinder. This proves Theorem 2.

5. Proof of Theorem (*). Let r denote the maximum rank of the second fundamental form of $X(M)$. Then by Appendix 1 and the assumption that M has at least one positive section curvature, r satisfies $2 \leqq r \leqq n$ and so $X(M)$ is not a hypercylinder. Let q be a point of M where the rank of the second fundamental form is r. Then the completeness of M and Lemma 2 of [8] or Lemma 2 of [4] can be used to show that there is a subset $L_q \subset M$ containing q and such that $X(L_q)$ is an $(n-r)$-flat. The argument which is essentially the same as that given at the beginning of the proof of Theorem III in [8], p. 913, will not be repeated here. By Theorem 2 above, M_1 is connected, hence Lemma 12 shows that every point p of M is in a subset $L_p \subset M$ such that $X(L_p)$ is an $(n-r)$-flat parallel to $X(L_q)$.

Let $(x^0, \cdots, x^n)$ be orthogonal coordinates in E^{n+1} such that $X(q)$ is the origin and $X(L_q)$ is the set $x^i = 0$ for $0 \leqq i \leqq r$. Let P_1 and P_2 denote respectively the orthogonal projections which map a point $(x^0, \cdots, x^n)$ of E^{n+1} to $(x^0, \cdots, x^r) \in E^{r+1}$ and $(x^{r+1}, \cdots, x^n) \in E^{n-r}$. At any point p of M the normal vector $N(p)$ is clearly orthogonal to $X(L_p)$. It follows easily that the image $P_1X(M)$ is an r-dimensional manifold M^* immersed in E^{r+1} where

$2 \leqq r \leqq n$. Let $X^*: M^* \to E^{r+1}$ be the immersion map. The connectedness of M_1 implies that M is locally convex under X, hence M^* is locally convex under X^* in the sense of Van Heijenoort [23]. Since the rank of the second fundamental form of $X(M)$ is r at q, M^* is absolutely convex at the point of M^* corresponding to q. The theorem of Van Heijenoort [23], p. 241, implies that $X^*(M^*)$ is the boundary of a convex body B^* in E^{r+1}. It follows that $X(M)$ is the boundary of the convex body $B^* \times E^{n-r}$ in E^{n+1}. This completes the proof of Theorem (*). The assertions of the supplement to Theorem (*) are now clear.

6. A remark on the proof of Theorem (*). In the proof of Theorem (*), it was shown that every point p of M was contained in a subset L_p of M such that the normal is constant on L_p and all of the sets $X(L_p)$ are parallel $(n-r)$-flats, where $2 \leqq r \leqq n$. Hartman and Nirenberg proved the corresponding fact for $0 \leqq r < 2$ in the proof of their Theorem III of [8]. One might suspect that a similar result would hold if the semi-definiteness of the second fundamental form were replaced by assumption that the Jacobian of the spherical image map has a rank $\leqq r$. An example will be given here to show that no such conjecture holds.

Let E^4 have coordinates (x^0, x^1, x^2, x^3) and consider the hypersurface defined by $x^0 = z(x^1, x^2, x^3) = x^1 \sin(x^3) + x^2 \cos(x^3)$. Then it can be verified that the rank of the spherical image map is two at every point of this hypersurface. The normal to the hypersurface is constant only on the lines ($=$ 1-flats) determined by $x^3 = \text{const.}$ and $x^1 \cos(x^3) + x^2 \sin(x^3) = \text{const.}$, but these lines are not parallel.

7. Applications. (a) *A theorem of Lichnerowicz.* Lichnerowicz [14], p. 221, has proved a theorem on the Betti numbers of an orientable n-manifold imbedded in E^{n+1} with a definite second fundamental form. A much stronger conclusion follows immediately from Theorem (*), provided the imbedding is smooth enough.

(b) *Hypersurfaces with a homeomorphic projection onto a hyperplane.* Suppose that a complete hypersurface in E^{n+1} can be represented by a function $x^0 = z(x^1, \cdots, x^n)$ where z is defined and of class C^{n+1} on the hyperplane $x^0 = 0$. Then if the Hessian matrix $(\partial^2 x / \partial x^i \partial x^j)$ is semi-definite and at at least one point is of rank greater than one, Theorem (*) implies that the hypersurface bounds a convex body. The function z will be convex and non-linear, hence z is not $o(r)$ as $r \to \infty$, where $r^2 = \sum_{i=1}^{n} (x^i)^2$. If $n = 2$, $x^1 = x$,

$x^2 = y$, this shows that if $z_{xx}z_{yy} - z_{xy}^2 \geqq 0$ and inequality holds at one point, then z is not $o(r)$ as $r \to \infty$. This result complements a theorem of S. Bernstein (cf. E. Hopf [10]) in which the same conclusion is drawn from $z_{xx}z_{yy} - z_{xy}^2 \leqq 0$ with inequality at one point.

(c) *The Rigidity of Surfaces.* First, note that a slightly stronger version of Theorem (*) has actually been proved. For, the assumption that the isometry X is of class C^{n+1} was only used to prove that $X(M)$ is of class C^{n+1}. It would have been sufficient to assume that the isometry is of class C^2 and $X(M)$ is of class C^{n+1} as a differentiable manifold.

In view of this remark, the proof of Theorem (*) has the following corollary.

COROLLARY 3. *Let S_1 be a C^2 n-hypersurface which bounds a convex body in E^{n+1} and is not isometric to E^n. Let S_2 be an n-hypersurface of class C^{n+1} which is a C^2 isometric immersion of S_1 in E^{n+1}. Then S_2 bounds a convex body in E^{n+1}. Then S_2 bounds a convex body in E^{n+1}.*

The statement of Corollary 3 has meaning even if S_2 and the isometry are only continuous. This raises the questions: For which k, $1 < k < n+1$ is Corollary 3 correct if S_2 is of class C^k rather than of class C^{n+1}? For which k, $0 < k < n+1$ is Corollary 3 correct if S_2 is of class C^k and the isometry is of class C^1? The analogous question is false if S_2 and the isometry are only continuous, since, for example, a cap can be cut from a sphere, inverted and replaced. It seems likely that Theorem (*) and Corollary 3 are correct if C^k replaces C^{n+1} for $k \geqq 2$. On the other hand, the possibility that the statements become false for $k = 1$ is suggested by the results of Kuiper [13] which show that if $n = 2$ imbeddings of class C^1 can have surprising properties.

Corollary 3 can be used to show that in the statements of some theorems, the requirement that a smooth surface be convex is superfluous. This point will be illustrated by a rigidity theorem of Pogorelov (cf. [18]).

Rigidity Theorem. Let S_1 be a 2-dimensional surface which bounds a convex body in E^3. Suppose S_1 has a spherical image 2π. Then, if S_2 is a convex surface isometric to S_1, S_2 is congruent to S_1.

If S_1 and S_2 are required to be of class C^2 and C^3 respectively and the isometry is of class C^2, it is not necessary to assume that S_2 is convex or even without self-intersections because these properties follow from Corollary 3.

Appendix 1. Sectional Curvature and the Second Fundamental Form.

The purpose of this section is to verify the proposition below which contains all of the assertions made above on the properties of the second fundamental form of a hypersurface which are determined intrinsically.

PROPOSITION. *Let M be a Riemannian n-manifold and $X: M \to E^{n+1}$ a C^2 isometric immersion of M. Let p be a point of M and let H be the matrix of coefficients of the second fundamental form of $X(M)$ at p. Let n_0, n_1, n_2 denote respectively the dimensions of the subspaces belonging to the zero, positive, and negative eigenvalues of H. If every sectional curvature is zero at p, then $n_0 \geqq n-1$. If at least one sectional curvature at p is not zero, then the sectional curvatures at p determine the numbers n_0, n_1, n_2 up to an interchange of n_1 and n_2.*

Note that the sectional curvatures are defined under the conditions of the proposition even though M may not be of class C^3 and the Riemannian-Christoffel tensor cannot be defined by the usual formulae; cf. [8], p. 912.

The proposition follows immediately from two lemmas which are stated below. Let $H = (h_{ij})$ be a real symmetric n by n matrix. Let V_n denote the space of all real n-vectors, and let $Rxyxy = (Hx, x)(Hy, y) - (Hx, y)^2$ for x, y in V_n. Let N denote the set of all n-vectors x such that $Rxyxy = 0$ for all y in V_n. Let N_0 denote the nullspace of H.

LEMMA 1A. $N \supset N_0$. *If* $V_n = N$, *then* $\dim N_0 \geqq n-1$. *If* $V_n \neq N$, *then* $N = N_0$.

Proof. The assertion $N \supset N_0$ follows immediately from the definition of $Rxyxy$. If $\dim N_0 < n-1$, there are two distinct unit orthogonal eigenvectors x, y of H which belong to the non-zero eigenvalues λ, μ of H. Then $Rxyxy = \lambda\mu \neq 0$, hence $N_0 \neq V_n$. Finally, suppose, if possible, that $V_n \neq N \neq N_0$. Then there must be a vector x in N orthogonal to N_0. In this case, H must be indefinite, for if H is semi-definite and $y \neq x$ is any vector orthogonal to N_0, the generalized Schwarz inequality gives $Rxyxy > 0$. Therefore H is indefinite and the subspaces N_1 and N_2 belonging respectively to the positive and negative eigenvalues of H are both non-empty.

Let $x = x_1 + x_2$, where x_i is in N_i. If x_1 and x_2 are both non-null, then $Rxyxy < 0$ for $y = x_1$. If x_1 is null, let y be any unit vector in N_1, and if x_2 is null, let y be any unit vector in N_2. In either case $Rxyxy < 0$. This shows that x is not in N which proves Lemma 1A.

If K is a subspace of the orthogonal complement of N having dimension

at least two, call K *definite* if for every pair of vectors x, y in K, $x \neq y$, $Rxyxy > 0$. Let $m = \operatorname{Max}\{\dim M : M \text{ is definite}\}$ (with $m = 0$ if there are no definite subspaces). Let $m_0 = \operatorname{Max}\{\dim N_1, \dim N_2\}$.

LEMMA 2A. *If $m = 0$, then $m_0 \leqq 1$. If $m \geqq 2$, $m = m_0$.*

Proof. If $m_0 \geqq 2$, $N = N_0$ by Lemma 1A, and N_1 and N_2 are definite subspaces. Therefore $m_0 \geqq 2$ implies $m \geqq m_0$. In particular if $m = 0$, then $m_0 \leqq 1$. If $m \geqq 2$, $N = N_0$ by Lemma 1A and again $m \geqq m_0$. Suppose that there is a definite subspace K such that $\dim K > \dim N_1$. Then K contains a vector $x \neq 0$ orthogonal to the projection of N_1 into K. Then x is in N_2. Similarly if $\dim K > \dim N_2$ there is a vector $y \neq 0$ in $K - N_2$. Then if $\dim K > m_0$ K contains non-null vectors x, y in N_2 and N_1 respectively. This implies $Rxyxy < 0$, hence K is not definite. This proves the last assertion of Lemma 2A.

Appendix 2. On the Extrema of the Cprvatures of a Surface.

1. The theorems of Hilbert and Weyl. Let S be a piece of two-dimensional surface of class C^2 in E^3. If p is a point on S, $k_1(p)$ and $k_2(p)$ will denote the principal curvatures of S at p, which are determined up to a factor ± 1. $H(p) = \frac{1}{2}(k_1 + k_2)$ and $K(p) = k_1 k_2$ will denote respectively the mean and Gaussian curvatures of S at p. S will be called *locally convex* if $K > 0$ everywhere on S and S has no self-intersections. In this case it will be supposed that the normal to S is directed in such a way that $H \geqq K^{\frac{1}{2}} > 0$ $k_1 \geqq k_2 > 0$. A function $f = f(p)$ defined on S will be said to have a *local maximum* [*minimum*] *at* p_0 if there is a neighborhood U of p_0 such that $f(p) \leqq f(p_0)$ $[f(p) \geqq f(p_0)]$ for all p in U.

This appendix is concerned with the assertions:

(H_n) *Let S be a locally convex piece of surface of class C^n $(n \geqq 2)$. Suppose k_1 has a local maximum and k_2 a local minimum at a point p_0 on S. Then, in a neighborhood of p_0, S is a part of the surface of a sphere.*

(W_n) *Let S be a locally convex piece of surface of class C^n $(n \geqq 2)$. Suppose H has a local maximum and K a local minimum at a point p_0 on S. Then, in a neighborhood of p_0, S is a part of the surface of a sphere.*

The assertion (H_4) is due to Hilbert [9], Anhang V, p. 238, although he did not explicitly formulate (H_4). The proof fails if $n < 4$ because the existence and continunity of the second derivatives of k_1 and k_2 are used. Weyl proved (W_4) in [24], p. 72; cf. Chern [4], p. 287, for another proof.

Again both proofs fail if $n < 4$. Actually, (W_n) follows from (H_n). In fact, if H has maximum and K has a minimum at a point p_0 on S, then $k_1 = H + (H^2 - K)^{\frac{1}{2}}$ has a maximum and $k_2 = K/k_1$ a minimum at p_0. Therefore, a counterexample to (W_n) for any n is also a counterexample to (H_n).

Hilbert employed his theorem (H_4) to prove the rigidity of the sphere, and Chern used (H_4) to prove that all "special" Weingarten surfaces are spheres. A theorem of Grotemeyer [6] follows from (W_4) just as Chern's theorem follows from (H_4). Both Chern's and Grotemeyer's theorems are now known to be correct for surfaces of class C^2; cf. Pogorelov [19] and Aleksandrov [1]. In view of this, it is somewhat surprising that, as will be shown by the examples in Section 2 below,

(*) *the assertions* (H_2), (H_3), *and* (W_2) *are false.*

It will remain undecided whether or not the assertion (W_3) is correct.

2. Counterexamples. The counterexamples to (H_3) and (W_2) are both surfaces defined by functions of the form

$$z(x,y) = +ax^2 + y^2 - w(x,y,\lambda) + bw(y,x,\lambda) \tag{1}$$

for $x^2 + y^2 < R_0^2$, where

$$w(x,y,\lambda) = \tfrac{1}{2}x^2(x^2+y^2)^\lambda$$

is of class C^2 if $0 < \lambda \leqq \frac{1}{2}$ and of class C^3 if $\lambda > \frac{1}{2}$. R_0, a, b, λ are positive constants which will be specified more precisely later.

The curvatures of S are given by the formulae

$$\begin{aligned} H &= \tfrac{1}{2}\{(1+q^2)r - 2pqs + (1+p^2)t\}/(1+p^2+q^2)^{\frac{3}{2}}, \\ K &= (rt - s^2)/(1+p^2+q^2)^2, \end{aligned} \tag{2}$$

and

$$k_1, k_2 = H \pm (H^2 - K)^{\frac{1}{2}} \tag{3}$$

where, as usual, $p = z_x$, $q = z_y$, $r = z_{xx}$, $s = z_{xy}$, $t = z_{yy}$. It follows easily from (1)-(3) that the curvatures H, K, k_1, and k_2 are positive for $x^2 + y^2 < R_0^2$ if R_0 is sufficiently small. In order to determine whether the functions defined in (2), (3) have maxima or minina at the origin, their partial derivatives with respect to ρ for small $\rho > 0$ will be calculated. Here, (ρ, θ) are polar coordinates in the (x,y) plane.

A simple calculation shows that

$$(4)\qquad p = 2a\rho\cos\theta + O(\rho^{2\lambda+1}), \qquad q = 2\rho\sin\theta + O(\rho^{2\lambda+1})$$

and

$$(5)\qquad r = 2a - \rho^{2\lambda} f_1(\theta,\lambda,b), \qquad s = O(\rho^{2\lambda}), \qquad t = 2 - \rho^{2\lambda} f_2(\theta,\lambda,b)$$

where the estimates hold as $\rho \to 0$ uniformly in θ, a, b, for $0 < a, b \leqq$ const., with λ fixed. The functions f_1 and f_2 are the trigonometric polynomials

$$(6)\qquad \begin{aligned} f_1(\theta,\lambda,b) &= 1 + 5\lambda\cos^2\theta + 2\lambda(\lambda-1)\cos^4\theta \\ &\qquad - b\lambda\sin^2\theta(1 + 2(\lambda-1)\cos^2\theta) \\ f_2(\theta,\lambda,b) &= \lambda\cos^2\theta(1 + 2(\lambda-1)\sin^2\theta) \\ &\qquad - b(1 + 5\lambda\sin^2\theta + 2\lambda(\lambda-1)\sin^4\theta). \end{aligned}$$

Also, for $\rho \neq 0$,

$$(7)\qquad p_\rho = 2a\cos\theta + O(\rho^{2\lambda}), \qquad q_\rho = 2\sin\theta + O(\rho^{2\lambda})$$

and

$$(8)\qquad r_\rho = -2\lambda\rho^{2\lambda-1} f_1(\theta,\lambda,b), \qquad s_\rho = O(\rho^{2\lambda-1}), \qquad t_\rho = -2\lambda\rho^{2\lambda-1} f_2(\theta,\lambda,b).$$

It is not difficult to see that, for $\rho \neq 0$, (2)-(8) imply

$$H_\rho = \tfrac{1}{2}(r_\rho + t_\rho) + O(\rho), \qquad K_\rho = 2(r_\rho + a t_\rho) + O(\rho) + O(\rho^{4\lambda-1})$$

and for $a > 1$

$$k_{1\rho} = r_\rho + O(\rho) + O(\rho^{4\lambda-1}), \qquad k_{2\rho} = t_\rho + O(\rho) + O(\rho^{4\lambda-1}).$$

Hence

$$(9)\qquad H_\rho = -\lambda\rho^{2\lambda-1}(f_1 + f_2) + O(\rho)$$

$$(10)\qquad K_\rho = -4\lambda\rho^{2\lambda-1}(f_1 + a f_2) + O(\rho) + O(\rho^{4\lambda-1})$$

$$(11)\qquad k_{1\rho} = -2\lambda\rho^{2\lambda-1} f_1 + O(\rho) + O(\rho^{4\lambda-1})$$

$$(12)\qquad k_{2\rho} = -2\lambda\rho^{2\lambda-1} f_2 + O(\rho) + O(\rho^{4\lambda-1}).$$

To obtain a counterexample to (W_2), let a, b be fixed, $0 < b < 1$, $ab > 1$. If $\lambda = 0$,

$$f_1 + f_2 \equiv 1 - b > 0 \text{ and } f_1 + a f_2 = 1 - ab < 0.$$

The forms of f_1, f_2 show that if $\lambda = \lambda(a,b)$ is sufficiently small, then

$$(13)\qquad f_1 + f_2 > 0 \text{ and } f_1 + a f_2 < 0 \text{ for all } \theta.$$

It can be supposed that $\lambda < 1$. Then (9), (10), and (13) show that H has a relative maximum and K a relative minimum at the origin. A more detailed

computation shows that (13) cannot hold unless $\lambda < \frac{1}{2}$, hence a counterexample to (W_3) cannot be found in this manner.

A counterexample to (H_3) is obtained by choosing $a > 1$ and b and λ such that $\lambda > \frac{1}{2}$ and

$$f_1 > 0 \text{ and } f_2 < 0 \text{ for all } \theta. \tag{14}$$

Such a choice is possible because if $\lambda = \frac{1}{2}$ and $\frac{1}{2} < b < 2$ then $f_1 > 1 - b/2 > 0$ and $f_2 < \frac{1}{2} - b < 0$ for all θ. Then (11), (12), and (14) show that k_1 has a relative maximum and k_2 a relative minimum at the origin. This shows that (H_3) is false.

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ON THE RATIONALITY OF THE ZETA FUNCTION OF AN ALGEBRAIC VARIETY.*

By Bernard Dwork.[1].

To Oscar Zariski on his sixtieth birthday.

Let p be a prime number, Ω the completion of the algebraic closure of the field of rational p-adic numbers and let $\mathfrak{K}$ be the residue class field of Ω. The field $\mathfrak{K}$ is the algebraic closure of its prime subfield and is of characteristic p. If T^* is the set of all roots of unity in Ω of order prime to p then the restriction of the residue class map to T^* is a multiplicative isomorphism of T^* onto the multiplicative group of $\mathfrak{K}$. The elements of $T = T^* \cup \{0\}$ form the Teichmüller representatives of $\mathfrak{K}$ in Ω and for each $x \in \mathfrak{K}$ the representative of x in Ω will be understood to be the element of T in the class x. The non-archimedean valuation of Ω will be denoted by the ordinal function, abbreviated "ord", and normalized by the condition $\operatorname{ord} p = 1$.

Let $\mathfrak{A}_n$ (resp: $\mathfrak{S}_n$) denote affine (resp: projective) space of dimension $n \geqq 1$ and characteristic p, viewed as consisting only of points with coordinates in $\mathfrak{K}$. Let k be the finite subfield of $\mathfrak{K}$ containing $q = p^a$ elements. A variety V in $\mathfrak{A}_n$ (resp: $\mathfrak{S}_n$) defined over k will be understood to be the subset consisting of the common zeros of a finite set of polynomials (resp: homogeneous polynomials) in n (resp: $n+1$) variables with coefficients in k. If V and W are varieties in $\mathfrak{A}_n$ (resp: $\mathfrak{S}_n$) defined over k then $V - W$ will be used to denote the set of points in V which are not in W and will be referred to as the difference between two varieties defined over k. Thus the empty subset of A_n (resp: S_n) is a variety and every variety is a difference between two varieties.

If V is the difference between two varieties defined over k, let N_s be the number of points of V with coordinates in the field of q^s elements in $\mathfrak{K}$. We define the zeta function of V to be the power series in one variable with rational coefficients:

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$$(1) \qquad \zeta(V;t) = \exp\{\sum_{s=1}^{\infty} N_s t^s/s\}.$$

It is an elementary consequence of Galois theory that the coefficients of this power series are integral. We note that while the expression "zeta function" has been used by other writers only in reference to non-singular, irreducible projective varieties, in our theory these restrictions play absolutely no role and therefore it serves no useful purpose in this discussion to restrict the expression in that way.

We have shown, [1], that the power series (1) represents an analytic function in the circle $\operatorname{ord} t > 0$ in Ω which has an analytic continuation to a meromorphic function in the region $\operatorname{ord} t > -\operatorname{ord} q$. This was shown to give some information about the zeta function in the classical sense (i. e., as a function on the complex plane), in particular that the first Betti number of a hypersurface of even dimension as defined by Weil [2] is certainly zero if the zeta function is rational.

In this paper we show that (1) always represents a rational function, thus generalizing and proving a part of Weil's conjecture [2]. This is done by showing (§ 3) that it represents a function having an analytic continuation to a meromorphic function on all of Ω. This gives rationality (§ 4) by arithmetic and function theoretic methods using the fact that the coefficients are rational integers and that the power series is (obviously) dominated in the archimedean sense by $(1 - q^n t)^{-1}$.

The geometrical part of our treatment rests upon the computation of the number of points of a hypersurface rational over a finite field. The use of additive and multiplicative characters for such computations is classical and is closely connected with the theory of Gauss sums. The methods of Weil [2] and Hua and Vandiver [3] are readily generalized to arbitrary hypersurfaces giving explicit formulae in terms of Gauss sums. We avoid the use of Gauss sums as the Hasse-Davenport relations [4, equ. (0.8)] do not give enough information for general hypersurfaces. These relations, which play a key role in Weil's treatment of special hypersurfaces, are replaced here by a group theoretical device introduced previously, [1], and reproduced here (§ 2) so as to make this paper relatively self contained. In [1] we used only multiplicative characters, in particular an approximation of the trivial multiplicative character of finite fields suggested by Warning [5]. We now use approximations of the additive characters of finite fields. The main difficulty is the determination of multiplicative relations between additive characters of different finite fields (of the same characteristic). This difficulty is overcome (§ 1) by the construction of an Ω valued function θ on $\mathfrak{K}$ (or equivalently

on T) which can be used to split a non-trivial character of the field of p^s elements into a product of s factors. This splitting together with the group theoretical methods of § 2 gives the analytic continuation of the zeta function on Ω. Some connections betwen the θ function and the theory of Gauss sums are noted (§ 1).

I am indebted to Professor Washnitzer for many discussion of this problem and for considerable encouragement.

Throughout this paper, Ω, $\mathfrak{K}$, k, q, p, T, ord are as above. Z is the ring of integers, Z_+ the set of non-negative integers, Q the field of rational numbers, Q' the p-adic completion of Q in Ω, $\mathfrak{O}$ the ring of integers in Ω, $\mathfrak{O}'$ the ring of integers in Q' and $\mathfrak{o}$ the ring of p-integral rational numbers (i.e., $\mathfrak{o} = \mathfrak{O}' \cap Q$). If $X = (X_1, \cdots, X_n)$ is a set of indeterminates, $b \in Z$, $u = (u_1, \cdots, u_n) \in (Z_+)^n$ then X^b denotes the set $X_1{}^b, X_2{}^b, \cdots, X_n{}^b$ while X^u denotes the monomial $\prod_{i=1}^{n} X_i{}^{u_i}$. *Monomial* will always be used in this sense, i.e., the coefficient of a monomial will always be understood to be 1. If $u = (u_1, \cdots, u_n) \in Z_+{}^n$ then *weight* u is defined to be the real number $\sum_{i=1}^{n} u_i$.

1. The splitting of additive characters. Let L be the maximal unramified extension of Q' in Ω, σ the Frobenius automorphism of L over Q' and $Y_1, Y_2, \cdots, Y_m$ a finite set of variables. If $f(Y) = f(Y_1, \cdots, Y_m)$ is a power series in these variables with coefficients in L, let $f^\sigma(Y)$ be the power series obtained by replacing the coefficients of $f(Y)$ by their images under σ, let $f(Y^p) = f(Y_1{}^p, Y_2{}^p, \cdots, Y_m{}^p)$ and let $f(0) = f(0, 0, \cdots, 0)$.

LEMMA 1. *If $f(Y)$ is a power series in $Y_1, Y_2, \cdots, Y_m$ with coefficients in L and $f(0) = 1$ then the coefficients of f are integers in L if and only if the coefficients of the power series $f^\sigma(Y^p)/(f(Y))^p$ (other than the constant term) are divisible by p in the ring of integers of L.*

We omit the proof since this generalization of our criterion [6, Lemma 1] for p-adic integrality of coefficients of power series in one variable is proven precisely as in the case of one variable. The lemma will be applied to a power series in two variables with coefficients in Q and there will be no further reference to either the field L or the Frobenius automorphism.

If m is any non-negative integer, let $h(m, t)$ denote the binomial coefficient, $h(m, t) = (m!)^{-1} \prod_{j=0}^{m-1} (t - j)$, a polynomial in t of degree m with coefficients in Q. Let $H(t, Y) = (1 + Y)^t = \sum_{m=0}^{\infty} h(m, t) Y^m$, an element of

$Q\{t,y\}$, the ring of power series in two variables with rational coefficients. Let

$$(2) \qquad F(t,Y)=H(t,Y)\prod_{j=1}^{\infty}H((t^{p^j}-t^{p^{j-1}})/p^j,Y^{p^j}),$$

an element of $Q\{t,Y\}$. From the definitions,

$$F(t,Y)^p/F(t^p,Y^p)$$
$$=(1+Y)^{pt}/(1+Y^p)^t=\exp\{t\log[(1+Y)^p/(1+Y^p)]\}.$$

It is well known that $\log(1+pt)\in pt\mathfrak{O}'\{t\}$ and that $\exp(pt)\in 1+pt\mathfrak{O}'\{t\}$ and thus we may conclude with the aid of the lemma since $(1+Y)^p/(1+Y^p)$ $\in 1+pY\mathfrak{o}\{Y\}$ that the coefficients of F are p-integral, i.e., $F\in\mathfrak{o}\{t,Y\}$.

Let $B_m(t)=\sum h(i_0,t)\prod_{j=1}^{m}h(i_j,(t^{p^j}-t^{p^{j-1}})/p^j)$, the sum being over all finite sequences $i_0,i_1,\cdots$ in Z_+ such that $i_0+pi_1+p^2i_2+\cdots=m$. Hence $B_m(t)$ is a sum of polynomials of degree m and therefore degree $B_m\leqq m$. It follows from the definitions that

$$(3) \qquad F(t,Y)=\sum_{m=0}^{\infty}B_m(t)Y^m.$$

We now see that

$$(4) \qquad F(t,Y)=\sum_{m=0}^{\infty}t^m\alpha_m(Y),$$

where $\alpha_m(Y)\in\mathfrak{o}\{Y\}$ and Y^m divides $\alpha_m(Y)$ in $\mathfrak{o}\{Y\}$. It is clear that as a power series in two variables $F(t,Y)$ converges in Ω for $\operatorname{ord} t\geqq 0$, $\operatorname{ord} Y>0$, that $\alpha_m(Y)$ converges under these conditions and that (3) and (4) converge to the same value. In particular, let $\lambda+1$ be a primitive p-th root of unity and let

$$(5) \qquad \theta(t)=F(t,\lambda)=\sum_{m=0}^{\infty}\beta_m t^m$$

where β_m (sometimes denoted $\beta(m)$) $=\alpha_m(\lambda)$. We note that $\operatorname{ord}\lambda=1/(p-1)$ and therefore

$$(6) \qquad \operatorname{ord}\beta_m\geqq m/(p-1),$$

an estimate of central importance in our computations.

For a fixed integer $s>1$, let t be a representative in T of an element t' in the field of p^s elements (hence $t^{p^s}=t$). It follows from (2), considering $F(t,Y)$ as a power series in Y with integral coefficients in $Q'(t)$, that

$$(7) \qquad H(t+t^p+\cdots+t^{p^{s-1}},Y)=\prod_{i=0}^{s-1}F(t^{p^i},Y).$$

It is well known (and may be verified by means of the lemma) that if $x \in \mathfrak{O}'$ then $(1+Y)^x \in \mathfrak{O}'\{Y\}$ and therefore converges for $|Y| < 1$. It is easily verified that if L is the unramified extension of Q' of degree s (so the residue field of L is the field of p^s elements) and if S_s is the trace of L over Q' then $x \to H(S_s(x), \lambda)$ is a non-trivial additive character of the ring of integers of L which is trivial on the maximal ideal and therefore gives, by passage to quotients, a non-trivial additive character of the residue class field of L. Hence $t' \to \Theta_s(t') = H(S_s(t), \lambda) = H(t + t^p + \cdots + t^{p^{s-1}}, \lambda)$ is a non-trivial additive character of the field of p^s elements. Thus equation (7) may be written in the form

$$H(S_s(t), Y) = \prod_{i=0}^{s-1} F(t^{p^i}, Y) \tag{7'}$$

and replacing Y by λ we obtain

$$\Theta_s(t') = H(S_s(t), \lambda) = \prod_{i=0}^{s-1} \theta(t^{p^i}). \tag{8}$$

Thus (8) gives a splitting of Θ_s mentioned in the introduction.

Note. The object of the above discussion was the demonstration of the existence of a power series, $\theta(t)$, satisfying condition (6) and (8). These properties do not characterize $\theta(t)$ and certain remarks of the referee have led us to a somewhat simpler construction of a power series satisfying these conditions. Let $E(X)$ denote the Artin-Hasse exponential series,

$$E(X) = \exp\{-\sum_{j=0}^{\infty} X^{p^j}/p^j\}.$$

There exists a unique element, η, in $Q'(\lambda)$ such that $E(\eta) = 1 + \lambda$. It is an elementary exericse to verify that the power series $E(\eta t)$ satisfies conditions (6) and (8). This function is by no means new since it has appeared in investigations of the norm residue symbol of Kummer extensions of algebraic number fields.

Although not needed for our subsequent discussion, we note that (8) may be applied to the theory of Gauss sums. Let

$$j = j_0 + pj_1 + \cdots + p^{s-1}j_{s-1} \in Z, \qquad 0 \leqq j_i \leqq p-1,$$

not all j_i equal to $p-1$. Let $g_s(j) = \sum t^{-j}\Theta_s(t')$, the sum being over the $p^s - 1$ roots of unity in T. Then $g_s(j)$ is, as is well known [4], the image in Ω of a Gauss sum on the field of p^s elements. Using (5) and (8) we see that

$$g_s(j) = (p^s - 1) \sum \prod_{j=0}^{s-1} \beta(i_j), \tag{9}$$

the sum being over all $(i_0, i_1, \cdots, i_{s-1}) \in Z_+^s$ such that

$$\sum_{e=0}^{s-1} i_e p^e \equiv j \bmod p^s - 1.$$

From (2) we see that

$$\beta(m) = (-p^{-1}\log(1+\lambda^p))^m/m! \qquad \text{for } 0 \leqq m < p \tag{10}$$

and hence $\operatorname{ord}\beta(m) = m/(p-1)$ for $0 \leqq m < p$. It now follows from (6) that

$$g_s(j)/\prod_{i=0}^{s-1}\beta(j_i)) \equiv -1 \bmod \lambda^{p-1}. \tag{11}$$

Thus letting $\sigma(j) = j_0 + j_1 + \cdots + j_{s-1}$, $\gamma(j) = j_0!\, j_1! \cdots j_{s-1}!$, and noting that $p^{-1}\log(1+\lambda^p) \in p^{-1}\lambda^p(1+(\lambda^{p-1}))$, we have

$$-g_s(j)/(-\lambda^p/p)^{\sigma(j)} \equiv \gamma(j)^{-1} \bmod \lambda^{p-1}. \tag{12}$$

Stickelberger's congruence [17],

$$-g_s(j)/\lambda^{\sigma(j)} \equiv \gamma(j)^{-1} \bmod \lambda \tag{12.1}$$

follows directly from (12) since

$$\lambda^{p-1}/(-p) \equiv 1 \bmod \lambda. \tag{12.2}$$

While (12) is ostensibly stronger than (12.1), it is in fact a consequence of (12.1) and the fact that $Q'(\lambda)$ is a Kummer extension of Q'. It follows from (12.2) that there exists, π, a root of $x^{p-1} + p = 0$ in $Q'(\lambda)$ such that $\lambda/\pi \equiv 1 \bmod \lambda$. Hence, letting $u = -g_s(j)/(\pi^{\sigma(j)}/\gamma(j))$, it follows from (12.1) that $u \equiv 1 \bmod \lambda$. If α is any automorphism of $Q'(\lambda)/Q'$ then $u^{1-\alpha}$ is a $p-1$ root of unity and also $u^{1-\alpha} \equiv 1 \bmod \lambda$. Hence $u^{1-\alpha} = 1$ which shows that $u \in Q' \cap (1+(\lambda)) = 1 + (p) \subset 1 + (\lambda^{p-1})$. Finally

$$\lambda^p/(-p\pi) = \lambda^p/\pi^p \in (1+(\lambda))^p \subset 1 + (\lambda^p).$$

Hence $u(-p\pi/\lambda^p)^{\sigma(j)} \equiv 1 \bmod \lambda^{p-1}$ which is equivalent to (12).

2. Linear transformations of polynomial rings. In this section we describe a group theoretical device discussed in greater detail in [1].

Let $L[X] = L[X_1, \cdots, X_n]$ be the ring of polynomials in n variables over an arbitrary field, L. Let ψ be the endomorphism of $L[X]$ (as L-module, not as ring) defined by

$$\psi(X^u) = \begin{cases} X^{u/q} & \text{if } q \mid u \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

for all $u \in Z_+^n$. (In this section q need not be a power of a prime. In the application (§ 3), $q = p^a$). For $H \in L[X]$, let $\psi \circ H$ denote the endomorphism $\xi \to \psi(H\xi)$ of $L[X]$. For each $m \in Z_+$, let L_m denote the finite dimensional subspace of $L[X]$ consisting of all polynomials of degree not greater than m and let $(\psi \circ H)_m$ be the restriction of $\psi \circ H$ to L_m. It is easily verified that for $m \geqq m_0 = (\text{degree}\, H)/(q-1)$,

(i) $(\psi \circ H)_m$ is an endomorphism of L_m

(ii) the "characteristic polynomial," $\det(I - t(\psi \circ H)_m)$ is independent of m

(iii) for each integer s, $s \geqq 1$, the trace, $\mathrm{Tr}((\psi \circ H)_m)^s$ is independent of m.

We are therefore able to define $\det(I - t(\psi \circ H))$ and $\mathrm{Tr}(\psi \circ H)^s$ in a natural way.

Let now L be algebraically closed and of characteristic zero. Let G_1 be the group of all roots of unity in L of order prime to q and let G be (for some fixed integer $n \geqq 1$) the n-fold direct product of G_1. Our technical device may now be stated:

LEMMA 2. *If $H \in L[X] = L[X_1, \cdots, X_n]$, $s \in Z$, $s \geqq 1$, then*

$$(q^s - 1)^n \,\mathrm{Tr}(\psi \circ H)^s = \sum H(x) H(x^q) \cdots H(x^{q^{s-1}}), \tag{14}$$

the sum being over all $x \in G$ such that $x^{q^s-1} = 1$.

Proof. Since $(\psi \circ H)^s = \psi^s \circ \{H(X) H(X^q) \cdots H(X^{q^{s-1}})\}$ and ψ^s is the endomorphism of $L[X]$ obtained by replacing q by q^s in the definition of ψ, we see that by taking a new value for q and a new choice for H we may assume $s = 1$. But $H \to \mathrm{Tr}(\psi \circ H)$ and $H \to \sum H(X)$ (the sum being over the elements of G of exponent $q-1$) are homomorphisms of $L[X]$, as L module, into L. Hence it may be assumed that H is a monomial and the verification becomes trivial: if $H = X^v$ then we find $\mathrm{Tr}(\psi \circ H) = 1$ if $(q-1) \mid v$ and is zero otherwise.

3. The meromorphic character of the zeta function. Let V be the difference between two varieties defined over k (in either $\mathfrak{A}_n$ or $\mathfrak{S}_n$). We now show that the analytic function in the circle $\mathrm{ord}\, t > 0$ in Ω represented by (1) has an analytic continuation to a meromorphic function on all of Ω, i. e., we show that the power series is the ratio of two power series in $1 + t\Omega\{t\}$ which converge for all $t \in \Omega$. While Krasner [8] has developed a theory of

analytic continuation in Ω, this will not be needed since the equivalent statements in terms of formal power series will be adequate for our purpose.

THEOREM 1. *$\zeta(V,t)$ is meromorphic.*

Proof. 1. To fix ideas let V be a difference between two varieties, say $V_1 - V_2$ which lie in $\mathfrak{A}_n$ (the projective case requires no more than changes in notation). Then $V = V_1 - V_1 \cap V_2$ and therefore

$$\zeta(V,t) = \zeta(V_1,t)/\zeta(V_1 \cap V_2, t)$$

Hence it is enough to prove the assertion for varieties defined over k, i.e., let $V = \bigcap_{i=1^r}^{r} V_i$, V_i a hypersurface defined over k. Let S_r be the set $\{1, 2, \cdots, r\}$ and for each non-empty subset, B, of S_r let $W_B = \bigcup_{i \in B} V_i$. Then W_B is a hypersurface and by an obvious combinatorial argument,

$$\zeta(V,t) = \prod \zeta(W_B,t)^{-(-1)^{m(B)}}, \tag{15}$$

where $m(B)$ is the number of elements in B and the product is over all subsets, B, of S_r. Hence it is enough to prove the assertion for hypersurfaces.

Let V be a hypersurface in $\mathfrak{A}_n$, let S be the set $\{1, 2, \cdots, n\}$. For each (proper or improper) subset B of S, let B' be the complementary subset of S, let W_B be the linear subvariety $\{X_i = 0\}_{i \in B}$ and let U_B be the degenerate hypersurface: $\prod_{i \in B} X_i = 0$. (If B is the empty subset of S, we understand U_B to be the empty subset of $\mathfrak{A}_n$ and W_B to be $\mathfrak{A}_n$). Clearly,

$$V = \cup (V \cap W_B - V \cap U_{B'}),$$

a disjoint union indexed by the subsets B of S. Hence it is enough to show that $Z(V - U_S, t)$ is meromorphic. This completes our reduction process.

2. Let $f(X) \in k[X] = k[X_1, \cdots, X_n]$ be the defining polynomial of a hypersurface V in $\mathfrak{A}_n$. Let V' be the degenerate hypersurface: $\prod_{i=1}^{n} X_i = 0$. We compute N_s, the number of points of $V - V'$ which are rational over k_s, the field of q^s elements in K.

Let Θ be a non-trivial additive character of k_s. Since $q = p^a$, we may take Θ to be Θ_{as} in the notation of § 1. For $u \in k_s$, $\sum \Theta(ux_0) = q^s$ if $u = 0$, zero otherwise, the sum being over all $x_0 \in k_s$. We write the sum in the form $1 + \sum \Theta(ux_0)$ the sum now being over the multiplicative group, k_s^* of k_s. Hence

$$q^s N_s = (q^s - 1)^n + \sum \Theta(x_0 f(x)) \tag{16}$$

the sum being over all $x \in k_s^{*n}$, $x_0 \in k_s^*$. It is now convenient to represent the polynomial $X_0 f(X) = X_0 f(X_1, \cdots, X_n)$ explicitly as a sum, $\sum_{i=1}^{\rho} A_i M_i$, where $A_1, A_2, \cdots, A_\rho$ are elements of k^* and $M_1, M_2, \cdots, M_\rho$ is a set of monomials in $n+1$ variables. Specifically, $M_i = X^{w_i}$ where $\{w_i\}_{i=1}^{\rho}$ is a set of ρ distinct elements of Z_+^{n+1}, it being understood that X now denotes the variables $X_0, X_1, \cdots, X_n$. We note (without further comment) that the first coefficient of each of the w_i is 1. Using the additive property of Θ, (16) becomes

$$q^s N_s = (q^s - 1)^n + \sum \prod_{i=1}^{\rho} \Theta(A_i M_i), \tag{17}$$

the sum being over all $x \in k_s^{*n+1}$.

For $i = 1, 2, \cdots, \rho$, A_i has a Teichmüller representative in Ω again denoted by A_i. Let T_s denote the $q^s - 1$ roots of unity in Ω, i.e., the Teichmüller representatives of k_s^*. Since $q = p^a$, it follows from § 1 that

$$q^s N_s = (q^s - 1)^n + \sum \prod_{i=1}^{\rho} \prod_{j=0}^{as-1} \theta((A_i M_i)^{p^j}) \tag{18}$$

the sum now being over the $(n+1)$-fold direct product T_s^{n+1} of T_s. Let $\Lambda(t) = \prod_{i=0}^{a-1} \theta(t^{p^i})$. Then $\Lambda(t) = \sum_{m=0}^{\infty} \lambda_m t^m$, where $\lambda_0 = 1$ and in general (writing $\beta(i)$ for β_i as in § 1), $\lambda_m = \sum \beta(i_0)\beta(i_1) \cdots \beta(i_{a-1})$, the sum being over all $(i_0, i_1, \cdots, i_{a-1}) \in Z_+^a$ such that $m = \sum_{j=0}^{a-1} i_j p^j$. To estimate $\operatorname{ord} \lambda_m$, we note that

$$\begin{aligned} \operatorname{ord}\Big(\prod_{j=0}^{a-1} \beta(i_j)\Big) &\geqq (i_0 + i_1 + \cdots + i_{a-1})/(p-1) \\ &\geqq mp/(q(p-1)) \geqq m/(q-1). \end{aligned}$$

Hence

$$\operatorname{ord} \lambda_m \geqq m/(q-1) \tag{6'}$$

and furthermore $\prod_{j=0}^{as-1} \theta(t^{p^j}) = \prod_{j=0}^{s-1} \Lambda(t^{q^j})$. Since $A_i^q = A_i$, $i = 1, 2, \cdots, \rho$, equation (18) assumes the form

$$q^s N_s = (q^s - 1)^m + \sum \prod_{i=1}^{\rho} \prod_{j=0}^{s-1} \Lambda(A_i M_i^{q^j}), \tag{18'}$$

the sum being as in (18).

It would be desirable to apply the methods of § 2 to the right side of (18′). Since no formulation of § 2 in terms of infinite series is available we proceed by means of p-adic approximations. For $r \in Z$, $r \geqq 1$, let

$$\Lambda_r(t) = \sum_{i=0}^{r(q-1)} \lambda_i t^i$$

so that for $\operatorname{ord} t \geqq 0$,

$$\Lambda_r(t) \equiv \Lambda(t) \bmod p^r.$$

Let

$$F_r(X) = \prod_{i=1}^{\rho} \Lambda_r(A_i M_i),$$

then

$$\prod_{j=0}^{s-1} F_r(X^{q^j}) = \prod_{i=1}^{\rho} \prod_{j=0}^{s-1} \Lambda_r(A_i M_i^{q^j}) \equiv \prod_{i=1}^{\rho} \prod_{j=0}^{s-1} \Lambda(A_i M_i^{q^j}) \bmod p^r.$$

Hence

$$q^s N_s \equiv (q^s - 1)^n + \sum \prod_{j=0}^{s-1} F_r(X^{q^j}) \bmod p^r, \tag{19}$$

the sum being as in (18). Thus from Lemma 2,

$$q^s N_s \equiv (q^s - 1)^n + (q^s - 1)^{n+1} \operatorname{Tr}(\psi \circ F_r)^s \bmod p^r. \tag{20}$$

Hence for each integer r greater than 1, there exists a sequence of elements of Ω, $\{b_{r,s}\}_{s=0}^{\infty}$ such that $\operatorname{ord} b_{r,s} \geqq r$ and such that

$$q^s N_s = (q^s - 1)^n + (q^s - 1)^{n+1} \operatorname{Tr}(\psi \circ F_r)^s + b_{r,s} \tag{20.1}$$

In the group $1 + t\Omega\{t\}$ of formal power series in one variable with coefficients in Ω and constant term 1, let the *weak topology* be the topology of pointwise convergence of coefficients, i. e., the multiplicative groups $V(m, \alpha) = 1 + t^m\Omega\{t\} + \alpha t\mathfrak{O}\{t\}$, $\alpha \in \mathfrak{O}$, $m \in Z_+$, $m > 0$, form a basis of the neighborhoods of 1. We note that $1 + t\mathfrak{O}\{t\}$ is a complete topological group under the weak topology. Let δ be the homomorphism $h(t) \to h(t)/h(tq)$ of $1 + t\Omega\{t\}$ into itself. Clearly $\delta^{-1}h(t) = \prod_{i=1}^{\infty} h(tq^i)$, the product being convergent in the weak topology, and so δ is a group automorphism of $1 + t\Omega\{t\}$. On the other hand δ and δ^{-1} map $V(m, \alpha)$ into itself for each $\alpha \in \mathfrak{O}$ and each integer, m, greater than zero. Hence δ is a topological group automorphism of $1 + t\Omega\{t\}$.

Let ϕ be the mapping $g(t) \to g(tq)$ of $\Omega\{t\}$ into itself. Clearly

$$\log g(t)^\delta = (1 - \phi) \log g(t)$$

where 1 denotes the identity mapping of $\Omega\{t\}$ into itself. Hence

$$\sum_{s=1}^{\infty} (q^s - 1)^n t^s / s = -(\phi - 1)^n \log(1 - t) = \log(1 - t)^{-(-\delta)^n}$$

and

$$\sum_{s=1}^{\infty} (q^s - 1)^{n+1} (t^s/s) \operatorname{Tr}(\psi \circ F_r)^s = -(\phi - 1)^{n+1} \log \det (I - t\psi \circ F_r)$$
$$= \log \det (I - t\psi \circ F_r)^{-(-\delta)^{n+1}}.$$

It follows from (20.1) and these purely formal (i.e. non-topological) properties of the δ operator that

$$(20.2) \qquad \zeta(V - V', qt) = (1-t)^{-(-\delta)^n} \det (I - t\psi \circ F_r)^{-(-\delta)^{n+1}} \exp\{\sum_{s=1}^{\infty} b_{r,s} t^s/s\}.$$

It is easily verified that for $\operatorname{ord} \alpha \geqq 1/(p-1)$, $\exp\{V(m,\alpha) - 1\} \subset V(m,\alpha)$, i.e., if we define the weak topology on $t\Omega\{t\}$ in the obvious way, then $g(t) \to \exp\{g(t)\}$ is a continuous map of $t\Omega\{t\}$ into $1 + t\Omega\{t\}$. In the weak topology of $t\Omega\{t\}$, we have $\lim_{r\to\infty} \sum_{s=1}^{\infty} b_{r,s} t^s/s = 0$ and hence

$$\lim_{r\to\infty} \exp\{\sum_{s=1}^{\infty} b_{r,s} t^s/s\} = 1.$$

We may now conclude from (20.2) and the topological group theoretic properties of the δ operator that $\Delta(t) = \lim_{r\to\infty} \det(I - t\psi \circ F_r)$ exists in our topology and that

$$(21) \qquad \zeta(V - V', qt) = (1-t)^{-(-\delta)^n} \Delta(t)^{-(-\delta)^{n+1}}.$$

Let $\det(I - t\psi \circ F_r) = \sum_{m=0}^{\infty} \gamma_{r,m} t^m$. We shall show that there exist non-negative real numbers, $z_1, z_2, \cdots$ such that $z_m \to \infty$ and $(\operatorname{ord} \gamma_{r,m})/m \geqq z_m$ for all $r, m \in Z_+$. It will then be clear that $\Delta(t) = \lim_{r\to\infty} \det(I - t\psi \circ F_r)$ is a power series in $\Omega\{t\}$ which converges for all $t \in \Omega$, i.e., is entire. This together with (21) will complete the proof of the theorem.

Let d be the degree of the (not necessarily homogeneous) polynomial f in n variables, defining the hypersurface V. We write $F_r(X) = \sum B_u X^u$, the sum being over some finite subset of Z_+^{n+1}. It follows from the definitions that

$$(22) \qquad B_u = \sum \prod_{i=1}^{\rho} \lambda(b_i) A_i^{b_i}$$

the sum being over all $(b_1, b_2, \cdots, b_\rho) \in Z_+^\rho$ such that

$$(23) \qquad \sum_{i=1}^{\rho} b_i w_i = u,$$

it being understood in (22) that $\lambda(m)$ denotes λ_m. Since weight $w_i \leqq d+1$ for $i = 1, 2, \cdots, \rho$, it follows from (23) that

$$\text{weight } u \leqq (d+1) \sum_{i=1}^{\rho} b_i \leqq (d+1)(q-1) \operatorname{ord}\left(\prod_{i=1}^{\rho} \lambda(b_i) A_i^{b_i} \right)$$

Hence

$$\operatorname{ord} B_u \geqq (\text{weight } u)/((d+1)(q-1)), \tag{24}$$

an estimate independent of r. For convenience let B_u also be denoted by $B(u)$. For fixed r, we may form a matrix representation, E, of $\psi \circ F_r$, indexed by a finite set of pairs (u, v) of elements of Z_+^{n+1}, by letting $E(u, v)$ be the coefficient of X^v in the polynomial $\psi(X^u F_r(X))$. Clearly $E(u, v) = B(qv - u)$. But $\gamma_{r,m}$ is the coefficient of t^m in the "characteristic equation" $\det(I - tE)$ of E and therefore is a sum and difference of products of the form

$$P = \prod_{i=1}^{m} E_r(u_i, v_i),$$

where $\{u_1, u_2, \cdots, u_m\}$ is a set of m *distinct* elements of Z_+^{n+1} and $\{v_1, v_2, \cdots, v_m\}$ is the same set in a possibly different order. Using (24) we obtain, $(d+1)(q-1) \operatorname{ord} P \geqq \sum_{i=1}^{m} \text{weight}(qv_i - u_i) = (q-1) \sum_{i=1}^{m} \text{weight } u_i$. There are only $\binom{n+d}{d}$ elements of Z_+^{n+1} of weight d. For each $m \in Z_+$ there exists a unique $x \in Z_+$ such that

$$m = D_m + \sum_{i=0}^{x} \binom{n+i}{i} \tag{25}$$

where $0 \leqq D_m < \binom{n+x+1}{x+1}$. We now have by the distinctness of the u_i, $(d+1) \operatorname{ord} P \geqq \sum_{i=0}^{x} i \binom{n+i}{i} + (x+1) D_m$ and therefore

$$(d+1) \operatorname{ord} \gamma_{r,m} \geqq \sum_{i=0}^{x} i \binom{n+i}{i} + (x+1) D_m, \tag{26}$$

an estimate which is independent of r. Let $z_m(d+1)m$ be the right side of (26), then z_m is independent of r, and $(\operatorname{ord} \gamma_{r,m})/m \geqq z_m$ for all $r, m \in Z_+$. We claim that $\lim_{m \to \infty} z_m = \infty$. Let $a_i = \binom{n+i}{i}$ and note that $a_i \leqq a_{i+1}$. Clearly $(d+1) z_m \leqq x + 1$ and hence

$$(d+1) z_m \geqq \left(\sum_{i=0}^{x} i a_i \right) / \sum_{i=0}^{x} a_i \geqq \sum_{i=[x/2]}^{x} i a_i / \sum_{i=0}^{x} a_i \geqq \tfrac{1}{2} \sum_{i=[x/2]}^{x} i a_i / \sum_{i=[x/2]}^{x} a_i \geqq [x/2]/2,$$

and furthermore equation (25) shows that $x \to \infty$ as $m \to \infty$. Hence $z_m \to \infty$ as asserted and this completes the proof of the theorem.

4. Borel's theorem. In this section we complete the proof of rationality by function theoretic methods. In addition to our previous conventions we introduce the following: $| \ |$ will be used to denote the ordinary absolute value of real and complex numbers, $| \ |_p$ will be used for the valuation in Ω, normalized so that $|p|_p = 1/p$. The determinant of an $m \times m$ matrix will be indicated by given a typical line $\| a_{i,1}, a_{i,2}, \cdots, a_{i,m} \|_{i=1}{}^m$.

We shall make considerable use of the methods of Borel [9]. In particular we shall use

(1) A power series with integral coefficients which is meromorphic in a circle of radius greater than one in the complex plane represents a rational function.

(2) If $F(t) = \sum_{s=0}^{\infty} A_s t^s$ is a formal power series with coefficients in any field, let

$$N_{s,m} = \| A_{s+j}, A_{s+j+1}, \cdots, A_{s+j+m} \|_{j=0}{}^m, \tag{27}$$

then $F(t)$ is certainly a ratio of polynomials if there exists an integer, m, such that $N_{s,m} = 0$ for all integers, s, which are large enough.

We pause to state some known facts about functions on Ω, [10]. Everything stated may be deduced easily from the theory of Newton polygons of power series. To the best of our knowledge no proof of this theory is available in the literature. To overcome this deficiency an exposition of this theory together with proofs of the next two propositions will be given in a future paper. For the present we shall state what is needed and indicate an alternate treatment adequate to complete the proof of rationality of the zeta function.

For each element b of the extended real line, $[-\infty, \infty)$ let

$$U_b = \{x \in \Omega \mid \operatorname{ord} x > b\}.$$

An element

$$F(t) = \sum_{s=0}^{\infty} A_s t^s$$

of $\Omega\{t\}$ which converges in U_b is said to represent an *analytic function* in that region. If F is the ratio F_1/F_2 of power series F_1, F_2 which converge in U_b then F is said to represent a *meromorphic function* on U_b. F is said to represent an *entire* function if it converges on $U_{-\infty} = \Omega$.

PROPOSITION 1. *If F converges in $U_{b'}$ and is never zero on U_b, where $-\infty \leqq b' < b < \infty$ then the power series $1/F$ converges in U_b.*

PROPOSITION 2. *If F converges in $U_{b'}$ and $-\infty \leqq b' < b < \infty$ then there*

exists a polynomial, $P(t)$, and an element, $H(t) \in \Omega\{t\}$, (both depending upon b) such that H converges in $U_{b'}$, is never zero in U_b and such that $F(t) = P(t)H(t)$.

The following direct consequence of these statements is needed for our generalization of Borel's Theorem (Theorems 2 and 3 below).

PROPOSITION 3. *If F represents a function meromorphic on $U_{b'}$ and $-\infty \leqq b' < b < \infty$ then there exists a polynomial, $P(t)$, depending upon b, such that the power series $P(t)F(t)$ converges in U_b.*

For the proof of Theorem 2 (and hence to complete the proof of rationality of the zeta function) it is enough to know Proposition 3 for the special case $-\infty = b'$. In this case the statement of Proposition 3 is a direct consequence of a theorem of Schnirelmann [11]:

A power series in one variable which converges everywhere in Ω (i. e. an entire function) must be of the form

$$\alpha t^m \prod_{i=1}^{\infty} (1 - \lambda_i t)$$

where $\alpha, \lambda_1, \lambda_2, \cdots \in \Omega$, $m \in Z_+$ and $\lim_{i \to \infty} \lambda_i = 0$.

We note that this theorem is itself an elementary consequence of the previously mentioned theory of Newton polygons.

THEOREM 2. *If $F(t) = \sum_{s=0}^{\infty} A_s t^s \in Z\{t\}$ converges in the complex plane in a circle of radius R and is meromorphic in Ω in the circle, $|t|_p < r$, where*

$$Rr > 1$$

then F represents a rational function.

Proof. In the following, the symbols R and r are used to denote the radii of circles properly contained by the circles in the statement of the theorem, but so chosen that the inequality, $Rr > 1$, remains valid.

In view of Borel's theorem, we may assume that $R \leqq 1$ and therefore $r > 1$. By hypothesis there exists a polynomial, $P(t) = 1 + a_1 t + \cdots + a_e t^e$, such that $P(t)F(t) = \sum_{s=0}^{\infty} B_s t^s$ converges in Ω for $|t|_p \leqq r$. Hence there exists a positive real number, M, such that

$$(28) \qquad |A_s| < M/R^s, \qquad |B_s|_p < M/r^s.$$

Clearly, $B_{s+e} = A_{s+e} + a_1 A_{s+e-1} + \cdots + a_e A_s$, and therefore for $m > e$ we have

$$(29) \qquad N_{s,m} = \| A_{s+j}, A_{s+j+1}, \cdots, A_{s+j+e-1}, B_{s+j+e}, \cdots, B_{s+j+m} \|_{j=0}{}^{m},$$

and since $| A_s |_p \leqq 1$, $r > 1$,

$$(30) \qquad | N_{s,m} |_p \leqq M^{m+1-e} / r^{s(m+1-e)}.$$

Using (27) and (28), we have

$$(31) \qquad | N_{s,m} | \leqq (m+1)!\, M^{m+1} / R^{(s+2m)(m+1)}.$$

By hypothesis we may choose a positive integer, m, so large that $R^{m+1} r^{m+1-e} > 1$. For this value of m, it follows from (30) and (31) that there exists $s_0 \in Z_+$ such that $| N_{s,m} |_p \, | N_{s,m} | < 1$ for $s > s_0$. Since $N_{s,m} \in Z$, we see that $N_{s,m}$ does not satisfy the product formula for valuations in Q and therefore must be zero for $s > s_0$. The rationality of F now follows from the second result of Borel.

This completes the proof of rationality of $\zeta(V, t)$, since in the notation of § 3 the power series is dominated in the ordinary sense by $(1 - q^n t)^{-1}$ and therefore has a non-zero radius of convergence in the complex plane.

A further generalization of Borel's theorem is worth noting. If L is an algebraic number field and $\mathfrak{p}$ is a prime of L (finite or infinite) let $\Omega_\mathfrak{p}$ be the completion of the algebraic closure of the completion of L at $\mathfrak{p}$. We normalize the valuation $| \ |_\mathfrak{p}$ of $\Omega_\mathfrak{p}$ so that the product formula, $\prod | \alpha |_\mathfrak{p} = 1$, holds for all non-zero elements, α, of L, the product being over all primes of L. In particular if $\mathfrak{p}$ is an infinite prime then $\Omega_\mathfrak{p}$ is the field of complex numbers and $| \ |_\mathfrak{p}$ is the ordinary (resp: square of the ordinary) absolute value in that field if $\mathfrak{p}$ is a real (resp: complex) prime of L.

THEOREM 3. *If L is an algebraic number field and $F(t) = \sum_{i=1}^{\infty} A_s t^s \in L\{t\}$, then F is rational if and only if there exists a finite set, S, of primes of L such that*

(i) For each $\mathfrak{p} \notin S$, $| A_s |_\mathfrak{p} \leqq 1$ for all non-negative integers s.

(ii) For each $\mathfrak{p} \in S$, $F(t)$ is meromorphic in $\Omega_\mathfrak{p}$ in a circle $| t |_\mathfrak{p} \leqq R_\mathfrak{p}$, where $\{R_\mathfrak{p}\}_{\mathfrak{p} \in S}$ is a set of positive real numbers satisfying the condition

$$(32) \qquad \prod_{\mathfrak{p} \in S} R_\mathfrak{p} > 1.$$

Proof. If F is rational then (ii) is certainly satisfied if S is any non-empty set of primes. On the other hand if F is a ratio of polynomials with coefficients in a field containing L then since $F \in L\{t\}$, it is a ratio $f(t)/g(t)$

of polynomials, f, g, with coefficients in L and hence with no loss in generality, it may be assumed that the coefficients of f and g are algebraic integers and that $g(0)\neq 0$. Hence the coefficients of F are integral at each finite prime of L which does not divide $g(0)$. Hence (i) is satisfied if we take S to be the set of all infinite primes of L and all prime divisors of $g(0)$. (In particular if L is the field of rational numbers then (i) is a consequence of Eisenstein's Theorem: If $F\in Q\{t\}$ and represents algebraic function then $F\in Z\{t/m\}$ for some integer m.)

To prove the "if" part of the theorem we repeat the argument of Theorem 2. With no loss in generality we may supppose that the infinite primes of L lie in S and that for each $\mathfrak{p}\in S$, F is meromorphic in a circle of radius *strictly greater* than $R_{\mathfrak{p}}$ and that inequality (32) is still valid. Since the radius of convergence of F is non-zero at each prime of L, there exists a real number $D>0$, such that for each prime $\mathfrak{p}\in S$, the radius of convergence of F at $\mathfrak{p}$ is strictly greater than D. If $\mathfrak{p}\in$ S then there exists a polynomial $P_{\mathfrak{p}}(t)=1+a_1t+\cdots+a_et^e$ such that $P_{\mathfrak{p}}(t)F(t)=\sum_{s=0}^{\infty}B_{s,\mathfrak{p}}t^s$ converges in $\Omega_{\mathfrak{p}}$ in a circle of radius strictly greater than $R_{\mathfrak{p}}$. The coefficients, $a_1,a_2,\cdots,a_e$ of $P_{\mathfrak{p}}(t)$ lie in $\Omega_{\mathfrak{p}}$ and depend upon $\mathfrak{p}$ but it may be assumed that e is independent of $\mathfrak{p}$, that is, $e\geqq\deg P_{\mathfrak{p}}$ for all $\mathfrak{p}\in S$. Since S is a finite set, there exists a real number M such that

$$(33)\qquad |B_{s,\mathfrak{p}}|_{\mathfrak{p}}\leqq M/R_{\mathfrak{p}}^{s},\qquad |A_s|_{\mathfrak{p}}\leqq M/D^s$$

for each $\mathfrak{p}\in S$ and each non-negative integer, s. For each $\mathfrak{p}\in S$ equation (29) is valid if B_s is replaced by $B_{s,\mathfrak{p}}$. Let $z=(m+e-1)e$ if $D<1$, $z=0$ if $D\geqq 1$. For $\mathfrak{p}\in S$, let $\mu(\mathfrak{p})=2m(m+1-e)$ if $R_{\mathfrak{p}}<1$, $\mu(\mathfrak{p})=0$ if $R_{\mathfrak{p}}\geqq 1$. It follows from (33) and (29) that for $\mathfrak{p}\in S$,

$$(34)\qquad |N_{s,m}|_{\mathfrak{p}}\leqq(m+1)!\,M^{m+1}/\{D^{se+z}R_{\mathfrak{p}}^{s(m+1-e)+\mu(\mathfrak{p})}\}$$

and therefore

$$(35)\qquad \prod_{\mathfrak{p}\in S}|N_{s,m}|_{\mathfrak{p}}\leqq G_m/(\prod_{\mathfrak{p}\in S}D^eR_{\mathfrak{p}}^{m+1-e})^s$$

where $G_m=\prod_{\mathfrak{p}\in S}\{(m+1)!\,M^{m+1}/D^zR_{\mathfrak{p}}^{\mu(\mathfrak{p})}\}$ is a real number depending upon m but independent of s. Using (32) we see that m may be chosen such that

$$\prod_{\mathfrak{p}\in S}\{D^eR_{\mathfrak{p}}^{m+1-e}\}>1.$$

Let m be so chosen, then $\prod_{\mathfrak{p}\in S}|N_{s,m}|_{\mathfrak{p}}<1$ for all s greater than some integer s_0. Since $|N_{s,m}|_{\mathfrak{p}}\leqq 1$ for $\mathfrak{p}\notin S$, it is clear that $N_{s\,m}$ does not satisfy the product

formula in L for $s > s_0$. The rationality of F follows from the criterion of Borel since $N_{s,m} = 0$ for $s > s_0$.

We note that Peterson [13] also considered generalizations of Borel's theorem to $L\{t\}$ but his results correspond to the case in which S is the set of infinite primes of L and hence could not be used to exploit the results of p-adic analysis.

5. Applications. We note some immediate consequences of our main result.

1. If V is an affine or projective hypersurface and if α is an algebraic integer, $\alpha \neq 1$, such that the geometric mean of the ordinary absolute magnitudes of the conjugates of α over Q is less than q, then α^{-1} cannot be a zero (resp: pole) of $\zeta(V, t)$ if V is of even (resp: odd) dimension. Furthermore $t = 1$ is a pole of $\zeta(V, t)$ if V is a projective of even dimension and is not a pole if V is affine of odd dimension. In particular, if V is an irreducible, non-singular, projective variety of even dimension then the first Betti number ([2]) of V is zero. This is a direct consequence of [1], where we proved these statements in the projective case using the hypothesis of rationality. The affine statement is obtained by an obvious modification of this earlier treatment.

2. Each abstract variety, V, defined over k has a finite covering consisting of affine varieties defined over k whose intersections are affine. The rationality of the zeta function [12] of V now follows by an obvious combinatorial argument. In particular, the projective case could be treated in this way as a consequence of the affine case.

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WHITEHEAD PRODUCTS AND THE COHOMOLOGY STRUCTURE OF PRINCIPAL FIBRE SPACES.*

By Franklin P. Peterson.

1. Introduction. Let (E, p, B) be a principal fibre space with fibre F [3]. Let $\mu\colon F \times E \to E$ be the operation of the fibre on the total space. If (E, p, B) is part of the Postnikov system of a space X, then there is a connection between homotopy operations in X (e.g. Whitehead products), and $\mu^*\colon H^*(E) \to H^*(F \times E)$ (see [1]). In this paper we show how to compute μ^* in certain cases and give some applications to computing Whitehead products.

2. Functional cohomology operations. In this section we recall the properties of functional (primary) cohomology operations that we will need.

Let $\theta \in H^q(\pi, n; G)$, $f\colon X \to Y$, and $u \in H^n(Y; \pi)$. Assume that $f^*(u) = 0$ and $\theta(u) = 0$. Then one can define

$$\theta_f(u) \in H^{q-1}(X; G)/f^*(H^{q-1}(Y; G)) + {}^1\theta(H^{n-1}(X; \pi)),$$

where ${}^1\theta \in H^{q-1}(\pi, n-1; G)$ is the suspension of θ (see [3] for details). One of the main properties these operations have is the following one. If $g\colon W \to X$, then

$$g^*\theta_f(u) = \theta_{fg}(u) \in H^{q-1}(W; G)/g^*f^*(H^{q-1}(Y; G)) + {}^1\theta(H^{n-1}(W; \pi)).$$

Furthermore, one can define functional cup products and functional cohomology operations coming from sums of stable operations and cup products. For example, let $\theta \in H^q(\pi, n; G)$, $f\colon X \to Y$, $y \in H^{q-n}(Y; \pi')$, and $u \in H^n(Y; \pi)$. Assume that $f^*(u) = 0$ and $\theta(u) + y \smile u = 0$, where there is a given coefficient pairing $\pi' \otimes \pi \to G$. Then one can define

$$(\theta + y^{\smile})_f(u) \in H^{q-1}(X; G)/f^*(H^{q-1}(Y; G)) + ({}^1\theta + f^*(y)^{\smile})(H^{n-1}(X; \pi)).$$

3. Principal fibre spaces. In this section we study principal fibre spaces and prove our main theorem.

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Let (E, p, B) be a principal fibre space with fibre F (see [3] for definition and elementary properties). Let $i\colon F \to E$ be the inclusion, $\nu\colon F \times F \to F$ the multiplication in F, $\mu\colon F \times E \to E$ the operation of F on E, and $\eta\colon F \times E \to E$ the projection on the second factor. The following diagrams are commutative.

$$\begin{array}{ccc} F \times F & \xrightarrow{\nu} & F \\ \downarrow {\scriptstyle 1\times i} & & \downarrow {\scriptstyle i} \\ F \times E & \xrightarrow{\mu} & E \end{array} \qquad\qquad \begin{array}{ccc} F \times E & \xrightarrow{\mu} & E \\ \downarrow {\scriptstyle \eta} & & \downarrow {\scriptstyle p} \\ E & \xrightarrow{p} & B \end{array}$$

Let $f_0 \in F$ be the unit for ν. Then $i_E\colon E \to F \times E$ defined by $i_E(e) = (f_0, e)$ is such that $\mu i_E \simeq$ identity. Also, $i_F\colon F \to F \times E$ defined by $i_F(f) = (f, i(f_0))$ is such that $\mu i_F \simeq i$.

Let $\pi = \pi_n(F)$, and assume that F is $(n-1)$-connected. Let $\iota \in H^n(F;\pi) \approx \mathrm{Hom}(\pi,\pi)$ denote the canonical generator. Let $w \in H^{n+1}(B;\pi)$ be the image of ι under transgression (i.e., w is the characteristic class of (E, p, B)).

Let $k \in H^q(E;\pi')$. We wish to study $\mu^*(k)$. It is well-known that the exact cohomology sequence of $(F \times E, F \vee E)$ gives rise to the following exact sequence:

$$0 \to H^q(F \times E, F \vee E;\pi') \xrightarrow{j^*} H^q(F \times E;\pi')$$
$$\xrightarrow{h} H^q(F;\pi') + H^q(E;\pi') \to 0,$$

where $h = i_F^* + i_E^*$. Thus

$$h(\mu^*(k)) = i_F^*\mu^*(k) + i_E^*\mu^*(k) = i^*(k) + k.$$

Hence we may write

$$\mu^*(k) = 1 \otimes k + i^*(k) \otimes 1 + j^*(x),$$

where $x \in H^q(F \times E, F \vee E;\pi')$.

Let $\theta \in H^{q+1}(\pi, n+1;\pi')$, $\psi \in H^{q-n}(B;G)$ be such that $\theta(w) + \psi \cup w = 0$, with a given coefficient pairing $G \otimes \pi \to \pi'$.

Theorem 1. *Let k be a representative of*

$$(\theta + \psi^{\cup})_p(w) \in H^q(E;\pi')/p^*(H^q(B;\pi')) + ({}^1\theta + p^*(\psi)^{\cup})(H^n(E;\pi)).$$

Then

$$\mu^*(k) = 1 \otimes k + {}^1\theta(\iota) \otimes 1 + (-1)^{(q-n)n} x' \otimes p^*(\psi),$$

where ${}^1\theta(x') = {}^1\theta(\iota)$.

Proof.
$$\begin{aligned}\mu^*(k) &= \mu^*((\theta+\psi^\cup)_p(w)) = (\theta+\psi^\cup)_{p\mu}(w)\\ &= (\theta+\psi^\cup)_{p\eta}(w) = \eta^*((\theta+\psi^\cup)_p(w))\\ &= 1\otimes k \in H^q(F\times E;\pi')/\eta^*p^*(H^q(B;\pi'))\\ &\quad + ({}^1\theta+\eta^*p^*(\psi)^\cup)(H^n(F\times E;\pi)).\end{aligned}$$

Hence

$$\begin{aligned}\mu^*(k) &= 1\otimes k + 1\otimes u + [{}^1\theta + (1\otimes p^*(\psi))^\cup](x'\otimes 1 + 1\otimes x'')\\ &= 1\otimes k + 1\otimes u + 1\otimes {}^1\theta(x'') + 1\otimes(p^*(\psi)^\cup x'') + {}^1\theta(x')\otimes 1\\ &\quad + (-1)^{(q-n)n}x'\otimes p^*(\psi).\end{aligned}$$

Apply h and we see that $u + {}^1\theta(x'') + p^*(\psi)^\cup x'' = 0$, and ${}^1\theta(x') = i^*(k)$. By Lemma III. 3. 2 of [2], we have that $i^*(k) = {}^1\theta(\iota)$. Thus we have shown that ${}^1\theta(x') = {}^1\theta(\iota)$ and that

$$\mu^*(k) = 1\otimes k + {}^1\theta(\iota)\otimes 1 + (-1)^{(q-n)n}x'\otimes p^*(\psi).$$

4. Whitehead products. In this section, we show how Theorem 1 enables us to compute some Whitehead products in a space X from knowledge of the Postnikov system of X.

Let $p\colon X^{(n)}\to X^{(n-1)}$ be a fibre space with fibre $K(\pi,n)$ (e.g. part of the Postnikov system of a space X). Let $k\in H^q(X^{(n)};\pi_{q-1})$, $q>n+1$, and consider the fibre space $\bar p\colon X\to X^{(n)}$ with $K(\pi_{q-1},q-1)$ as fibre and k as k-invariant. As in Section 3, we may write $\mu^*(k) = 1\otimes k + i^*(k)\otimes 1 + j^*(x)$, where $x\in H^q(K(\pi,n)\times X^{(n)}, K(\pi,n)\vee X^{(n)};\pi_{q-1})$. Let

$$\alpha\in\pi_n(X)\approx\pi_n(K(\pi,n)),\qquad \beta\in\pi_{q-n}(X),\qquad \nu\colon\pi_i(X)\to H_i(X)$$

be the Hurewicz homomorphism, and let χ be the composition

$$\begin{aligned}&H^q(K(\pi,n)\times X^{(n)}, K(\pi,n)\vee X^{(n)};\pi_{q-1})\\ &\quad\to \mathrm{Hom}(H_q(K(\pi,n)\times X^{(n)}, K(\pi,n)\vee X^{(n)}),\pi_{q-1})\\ &\quad\to \mathrm{Hom}(H_n(K(\pi,n),\text{pt.})\otimes H_{q-n}(X^{(n)},\text{pt.}),\pi_{q-1}).\end{aligned}$$

Meyer [1] has proven the following theorem.

THEOREM 2. $[\alpha,\beta] = \chi(x)\ (\nu(\alpha)\otimes\bar p_*\nu(\beta))\in\pi_{q-1}(X)$.

Let $k^{n+1}\in H^{n+1}(X^{(n-1)};\pi)$ be the k-invariant for the fibre space $p\colon X^{(n)}\to X^{(n-1)}$. Let $\theta\in H^{q+1}(\pi,n+1;\pi')$, $\psi\in H^{q-n}(X^{(n-1)};G)$ be such that $\theta(k^{n+1}) + \psi^\cup k^{n+1} = 0$ with a given coefficient pairing $G\otimes\pi\to\pi_{q-1}$, and such that $k\in H^q(X^{(n)};\pi_{q-1})$ is a representative of $(\theta+\psi^\cup)_p(k^{n+1})$. [This con-

dition is many times fulfilled when $i^*(k) = {}^1\theta(\iota)$; e. g. see the example below.] We may then apply Theorems 1 and 2 to deduce the following theorem.

THEOREM 3. *$[\alpha,\beta] = (-1)^{(q-n)n} x'(\nu(\alpha)) \cdot \psi(p_* \bar{p}_* \nu(\beta)) \in \pi_{q-1}(X)$, where the pairing is the given coefficient pairing $\pi \otimes G \to \pi_{q-1}$, and where ${}^1\theta(x') = {}^1\theta(\iota)$.*

We give the following example to illustrate the applications of Theorem 3. The result here is known (see [4]); however, our method has the advantage of being purely cohomological in nature. Also, the method gives further insight into the structure of the spectral sequence of a fibre space.

Let CP^m be complex projective m-space. We shall calculate the Whitehead product pairing

$$\pi_{2m+1}(CP^m) \otimes \pi_2(CP^m) \to \pi_{2m+2}(CP^m).$$

(Recall that $\pi_{2m+1}(CP^m) = Z = \pi_2(CP^m)$ and $\pi_{2m+2}(CP^m) = Z_2$.) Let $\bar{\iota} \in H^2(Z,2;Z)$, then $\bar{\iota}^{m+1} \in H^{2m+2}(Z,2;Z)$ is the first k-invariant. To study the above Whitehead product in CP^m, it is enough to study it in X, the Postnikov system of CP^m through dimension $2m+2$. I.e., $\bar{p}: X \to X^{(2m+1)}$ and $p: X^{(2m+1)} \to K(Z,2)$, the first with fibre $K(Z_2, 2m+2)$ and k-invariant the non-zero class in $H^{2m+3}(X^{(2m+1)}; Z_2) = Z_2$, where $i^*(k) = \mathrm{Sq}^2(\iota)$, and the second with fibre $K(Z, 2m+1)$ and k-invariant $\bar{\iota}^{m+1}$. (For details of this computation, see [2].) In case m is odd, $\mathrm{Sq}^2(\bar{\iota}^{m+1}) = 0$. In case m is even, $\mathrm{Sq}^2(\bar{\iota}^{m+1}) + \bar{\iota} \cup \iota^{m+1} = 0 \bmod 2$, with the non-zero pairing $Z \otimes Z \to Z_2$. Thus we may apply Theorem 3 with $\psi = 0$ or $\bar{\iota}$ respectively. Since $\mathrm{Sq}^2(x') = \mathrm{Sq}^2(\iota) \neq 0$ in $H^{2m+3}(Z, 2m+1; Z_2)$, x' is an odd multiple of ι. If α is a generator of $\pi_{2m+1}(X) = \pi_{2m+1}(CP^m)$, and β a generator of $\pi_2(X)$, then $[\alpha,\beta] = 0$, if m is odd and, if m is even, $[\alpha,\beta] = \iota(\nu(\alpha)) \cdot \bar{\iota}(p_* \bar{p}_* \nu(\beta))$ = non-zero element of $\pi_{2m+2}(X) = Z_2$ as $\nu(\alpha)$ is dual to ι and $p_* \bar{p}_* \nu(\beta)$ is dual to $\bar{\iota}$.

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ON THE PREPARATION OF MANUSCRIPTS.

The following instructions are suggested or dictated by the necessities of the technical production of the American Journal of Mathematics. Authors are urged to comply with these instructions, which have been prepared in their interests.

Manuscripts not complying with the standards usually have to be returned to the authors for typographic explanation or revisions and the resulting delay often necessitates the deferment of the publication of the paper to a later issue of the Journal.

Horizontal fraction signs should be avoided. Instead of them use either solidus signs / or negative exponents.

Neither a solidus nor a negative exponent is needed in the symbols $\frac{1}{2}$, $\frac{1}{\pi}$, $\frac{1}{2\pi}$, $\frac{1}{2\pi i}$, which are available in regular size type.

Binomial coefficients should be denoted by $C_k{}^n$ and not by parentheses. Correspondingly, for symbols of the type of a quadratic residue character the use of some non-vertical arrangement is usually imperative.

For square roots use either the exponent $\frac{1}{2}$ or the sign $\surd$ without the top line, as in $\surd -1$ or $\surd(a+b)$.

Replace $e^{(\;)}$ by $\exp(\;)$ if the expression in the parenthesis is complicated.

By an appropriate choice of notations, avoid unnecessary displays.

Simple formulae, such as $A + iB = \frac{1}{2}C^n$ or $s_n = a_1 + \cdots + a_n$, should not be displayed (unless they need a formula number).

Use $'$ or $d\ /dx$, possibly D, but preferably not a dot, in order to denote ordinary differentiation and, as far as possible, a subscript in order to denote partial differentiation (when the symbol ∂ cannot be avoided, it should be used as $\partial\ /\partial x$).

Commas between indices are usually superfluous and should be avoided if possible.

In a determinant use a notation which reduces it to the form $\det a_{ik}$.

Subscripts and superscripts cannot be printed in the same vertical column, hence the manuscript should be clear on whether $a_i{}^k$ or $a^k{}_i$ is preferred. (Correspondingly, the limits of summation must not be typed *after* the Σ-sign, unless either $\Sigma_1{}^m$ or $\Sigma^m{}_1$ is desired.) If a letter carrying a subscript has a prime, indicate whether $x_i{}'$ or $x'{}_i$ is desired.

Experience shows that a tilde or anything else *over* a letter is very unsatisfactory. Such symbols often drop out of the type after proof-reading and, when they do not, they usually appear uneven in print. For these reasons we advise against their use. This advice applies also to a bar over a Greek or German letter (for the symbol of complex conjugation an asterisk is often allowed by the context). Type carrying bars over ordinary size italic letters of the Latin alphabet is available.

Bars reaching over several letters should in any case be avoided (in particular, type lim sup and lim inf instead of lim with upper and lower bars).

Repeated subscripts and superscripts should be used only when they cannot be avoided, since the index of the principal index usually appears about as large as the principal index. Bars and other devices *over* indices cannot be supplied. On the other hand, an asterisk or a prime (to be printed *after* the subscript) is possible on a subscript. The same holds true for superscripts.

Distinguish carefully between l. c. "oh," cap. "oh" and zero. One way of distinguishing them is by underlining one or two of them in different colors and explaining the meaning of the colors.

Distinguish between ϵ (epsilon) and $\in$ or ε (symbol), between x (eks) and $\times$ (multiplication sign), between l. c. and cap. phi, between l. c. and cap. psi, between l. c. k and kappa and between "ell" and "one" (for the latter, use l and 1 respectively).

Avoid unnecessary footnotes. For instance, references can be incorporated into the text (parenthetically, when necessary) by quoting the number in the bibliographic list, which appear at the end of the paper. Thus: "[3], pp. 261-266."

Except when informality in referring to papers or books is called for by the context, the following form is preferred:

[3] O. K. Blank, "Zur Theorie des Untermengenraumes der abstrakten Leermenge," *Bulletin de la Societé Philharmonique de Zanzibar*, vol. 26 (1891), pp. 242-270.

In any case, the references should be precise, unambiguous and intelligible.

Usually sections numbers and section titles are printed in bold face, the titles "Theorem," "Lemma" and "Corollary" are in caps and small caps, "Proof," "Remark" and "Definition" are in italics. This (or a corresponding preference) should be marked in the manuscript. Use a period, and *not* a colon, after the titles Theorem, Lemma, etc.

German, script and bold face letters should be underlined in various colors and the meaning of the colors should be explained. The same device is needed for Greek letters if there is a chance of ambiguity. In general, mark all cap. Greek letters.

All instructions and explanations for the printer can conveniently be collected on a separate sheet, to be attached to the manuscript.

In case of doubt, recent issues of the Journal may be consulted.